Séminaire BOURBAKI Juin 2010
62ème année, 2009-2010, n$^{\text{o}}$ 1022

FUNDAMENTAL GROUPS OF KÄHLER MANIFOLDS AND GEOMETRIC GROUP THEORY

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INTRODUCTION

The aim of this note is to report on some recent progress in the problem of characterizing fundamental groups of compact Kähler manifolds, henceforth called Kähler groups. More precisely we will illustrate, by means of a specific result, the program outlined by Delzant and Gromov in [DG05]: "Identify the constraints imposed by the Kähler nature of the space on the asymptotic invariants of its fundamental group and then express these invariants in terms of algebraic properties".

The result we have in mind is the theorem of T. Delzant [Del10] which says that a solvable Kähler group contains a nilpotent subgroup of finite index. This is based on the explicit description of the Bieri–Neumann–Strebel invariant of a Kähler group $\pi_1(M)$ in terms of factorizations of M over hyperbolic Riemann surfaces.

Before we come to this main topic we will recall what a Kähler manifold is, then list in telegraphic style results giving restrictions on Kähler groups and give a series of examples. For a more complete account of the theory of Kähler groups up to 1995, see [ABC+96].

Let M be a complex manifold with a Hermitian metric h, that is a collection of Hermitian metrics h_x on each tangent space T_xM, varying smoothly with x. Then the real part $g := \Re h$ gives a Riemannian metric on the underlying real manifold and the imaginary part $\omega := \Im h$ gives a real two-form. Together with the complex structure J we have

$$\omega(X, Y) = g(X, JY)\,. \tag{1}$$

The Hermitian manifold (M, h) is Kähler if $d\omega = 0$. An elementary consequence of this relation is that at each point of M there exist holomorphic coordinates such that the Hermitian metric equals the flat metric on $\mathbf{C}^n$ up to and including terms of first order, [Voi02, 3.14]. This readily implies the Kähler identities ([Voi02, 6.1]) which are at the basis of the Hodge decomposition of the cohomology of compact Kähler manifolds. The compatibility condition (1) implies that $\omega^n = n!d\,\mathrm{vol}_g$, in particular ω is non-degenerate at each point, i. e. it is a symplectic form and, when M is compact, defines therefore a non-zero class in $\mathrm{H}^2(M, \mathbf{R})$.

The following two observations lead to an important class of Kähler manifolds:

– the induced Hermitian structure on a complex submanifold $N \subset M$ of a Kähler manifold is Kähler;
– up to a positive multiple, there is a unique $\mathrm{SU}(n+1)$-invariant Hermitian metric on $\mathbf{CP}^n$; since its imaginary part ω is an invariant two-form, it is closed. Normalizing the metric so that $\int_{\mathbf{CP}^1} \omega = 1$, one obtains the Fubini–Study metric.

Thus every smooth projective manifold is a Kähler manifold. In our context this leads to the question whether every Kähler group is also the fundamental group of a smooth projective variety, to which we do not know the answer. Remarkably, concerning homotopy type, we have, thanks to Voisin [Voi04], examples of compact Kähler manifolds which do not have the homotopy type of a smooth projective variety. Finally, it is a natural question whether the existence of a complex structure and/or a symplectic structure on a compact manifold imposes additional restrictions on its fundamental group, beyond being finitely presentable. In fact, every finitely presentable group is the fundamental group of a complex threefold which is also symplectic ([Gom95], see also [MS98, 7.2]); it is thus the compatibility between these two structures, that is the defining property of a Kähler structure, which will give restrictions on its fundamental group.

1. RESTRICTIONS

In this section $\Gamma = \pi_1(M)$ is the fundamental group of a compact Kähler manifold M with Kähler form ω.

1.1. The first Betti number $b_1(\Gamma)$ is even

The vector space $\mathrm{Hom}(\Gamma, \mathbf{R}) = \mathrm{H}^1(\Gamma, \mathbf{R})$ is isomorphic to the space $\mathcal{H}^1(M)$ of real harmonic 1-forms on M; precomposition of 1-forms with J gives a complex structure on $\mathcal{H}^1(M)$ and hence its dimension $b_1(\Gamma)$ is even.

1.2. There is a non-degenerate skew-structure on $\mathrm{H}^1(\Gamma, \mathbf{R})$

On $\mathrm{H}^1(\Gamma, \mathbf{R})$ the form $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \wedge \omega^{n-1}$ is skew-symmetric and non-degenerate (Hard Lefschetz Theorem). Noting that the classifying map $M \to B\Gamma$ induces in cohomology an isomorphism in degree 1 and an injection in degree 2 shows that this skew-symmetric form factors through the cup product $\Lambda^2 \mathrm{H}^1(\Gamma, \mathbf{R}) \to \mathrm{H}^2(\Gamma, \mathbf{R})$, which is therefore not zero if $b_1(\Gamma) > 0$, [JR87]. In this context there is a conjecture of Carlson and Toledo, namely that if Γ is infinite, there is $\Gamma' < \Gamma$ of finite index with $b_2(\Gamma') > 0$. For more on this, see [Kol95, 18.16], [Klia], [Klib], [KKM].

1.3. The Malcev Lie algebra $\mathcal{L}\Gamma$ of Γ is quadratically presented

Associated to Γ there is a tower of nilpotent Lie algebras

$$\dots\longrightarrow\mathcal{L}_n\Gamma\longrightarrow\mathcal{L}_{n-1}\Gamma\longrightarrow\dots,$$

say over $\mathbf{R}$, where $\mathcal{L}_n\Gamma$ is the Lie algebra of the $\mathbf{R}$-unipotent algebraic group determined by the quotient $\Gamma/\mathcal{C}^n\Gamma$, where $\mathcal{C}^n\Gamma$ is the n-term of the descending central series. "Quadratic presentation" then means loosely that this tower of Lie algebras is determined by the map $\mathrm{H}_2(\Gamma)\to\Lambda^2\mathrm{H}_1(\Gamma)$ (see [ABC+96, Chap. 3] and references therein).

1.4. A Kähler group has zero or one end [(1)]

The ideas and methods introduced by Gromov [Gro89] leading to this result have been very influential in this field in the last twenty years. Here are some highlights. Recall that for the number $e(\Gamma)$ of ends of a finitely generated group we have $e(\Gamma)\in\{0,1,2,\infty\}$, with $e(\Gamma)=0$ precisely when Γ is finite and $e(\Gamma)=2$ precisely when Γ is virtually $\mathbf{Z}$; then Stallings' theorem says that $e(\Gamma)=\infty$ precisely when Γ is a nontrivial amalgam or an HNN-extension, both over a finite group. This theorem will however not be used in the proofs. The first step, which has nothing to do with Kähler manifolds, is the following

Proposition 1.1. — *If $\Gamma=\pi_1(M)$, where M is a compact Riemannian manifold and $e(\Gamma)=+\infty$, then the space $\mathcal{H}^1_{(2)}(\widetilde{M})$ of square integrable harmonic 1-forms on $\widetilde{M}$ is non-trivial, and in fact infinite dimensional. In particular, the reduced L^2-cohomology group $\overline{\mathrm{H}}^1\big(\Gamma,\ell^2(\Gamma)\big)$ does not vanish, as it is isomorphic to $\mathcal{H}^1_{(2)}(\widetilde{M})$ by a variant of Dodziuk's de Rham theorem.*

The central result is then the following factorization theorem:

Theorem 1.2 ([ABR92]). — *Let X be a complete Kähler manifold with bounded geometry and $\mathrm{H}^1(X,\mathbf{R})=0$. Assume that $\mathcal{H}^1_{(2)}(X)\neq 0$. Then there exists a proper holomorphic map with connected fibers $h:X\to\mathbb{D}$ to the Poincaré disk; moreover the fibers of h are permuted by* $\mathrm{Aut}(X)$.

We obtain then the following purely group theoretical consequence:

Corollary 1.3 ([ABR92], [Gro89]). — *Let Γ be a Kähler group with $\overline{\mathrm{H}}^1\big(\Gamma,\ell^2(\Gamma)\big)\neq 0$. Then Γ is commensurable to the fundamental group Γ_g of a compact orientable surface of genus $g\geq 2$.*

More precisely there are a subgroup $\Gamma'<\Gamma$ of finite index and an exact sequence

$$1\longrightarrow F\longrightarrow\Gamma'\longrightarrow\Gamma_g\longrightarrow 1$$

with F finite. In particular $e(\Gamma)=e(\Gamma')=e(\Gamma_g)=1$ and thus a Kähler group has zero or one end.

1. A general reference for this section is [ABC+96, Ch. 4]

The factorization Theorem 1.2 follows from a general stability theorem for compact leaves in singular holomorphic foliations, which also plays a central role in the work of Delzant and Gromov on “Cuts in Kähler groups”, [DG05] (see also § 1.5). Recall that the singular holomorphic foliation $\mathcal{F}_\eta$ associated to a closed holomorphic 1-form η on a complex manifold X is generated by the relations $x \sim_U y$, where U is an open set on which $\eta = df$ with f holomorphic and x, y are in U and are in the same connected component of a fiber of f.

Theorem 1.4 ([DG05, 4.1]). — *Let X be a complete Kähler manifold of bounded geometry, η a closed holomorphic 1-form on X and $\mathcal{F}_\eta$ the associated singular holomorphic foliation. If $\mathcal{F}_\eta$ has one compact leaf, all leaves are compact.*

One important principle here, which is an immediate consequence of the volume monotonicity property of analytic subsets of $\mathbf{C}^n$ leading to the definition of Lelong numbers [Chi89, 15.1, Prop. 1], is the following uniform boundedness property of submanifolds of finite volume.

Proposition 1.5. — *If X is Kähler, complete and of bounded geometry, then for every $T > 0$ and $\epsilon > 0$ there is $N(T, \epsilon) \in \mathbf{N}$ such that every closed (as a subset of Y) complex submanifold $Y \subset X$ with $vol(Y) \leq T$ can be covered by $N(T, \epsilon)$ balls of radius ϵ. In particular Y is compact.*

The proof of Theorem 1.2 then proceeds as follows: let $\alpha \in \mathcal{H}^1_{(2)}(X)$ and η_α be the L^2-holomorphic 1-form with $\alpha = \Re\eta_\alpha$. Let $f : X \to \mathbf{C}$ be holomorphic with $df = \eta_\alpha$; the co-area formula together with the L^2-condition implies that f has a fiber of finite volume. This implies by the above fact that $\mathcal{F}_{\eta_\alpha}$ has a compact leaf and by Theorem 1.4 that all leaves are compact, so that one can apply Stein factorization. The final point consists in showing that $\mathcal{F}_{\eta_\alpha}$ does not depend on the particular choice of α; this follows from a tricky argument in L^2-Hodge theory, using the boundedness of η_α ([Gro91] or [ABC$^+$96, lemma 4.16]) which gives that $\eta_\alpha \wedge \eta_\beta = 0$ for any choice $\alpha, \beta \in \mathcal{H}^1_{(2)}(X)$, and hence $\mathcal{F}_{\eta_\alpha} = \mathcal{F}_{\eta_\beta}$.

1.5. A Kähler group with at least three relative (stable) ends “fibers”

We have seen that if M is compact Kähler, then the number $e(\widetilde{M})$ of ends of the universal covering $\widetilde{M}$ is 0 or 1. By taking M to be a Riemann surface of genus $g \geq 2$, one sees that the number of ends $e(X)$ of an arbitrary covering $X \to M$ can take any value in $\mathbf{N} \cup \{\infty\}$. This leads naturally to the question whether the existence of a many ended covering $X \to M$ of a compact Kähler manifold imposes restrictions on its fundamental group Γ. An answer is given by Delzant and Gromov under a stability condition: let $\Lambda < \Gamma$ be the subgroup corresponding to X; then the Γ-space Γ/Λ is stable if Γ/Λ is infinite and $\mathrm{H}^1\big(\Gamma, \ell^2(\Gamma/\Lambda)\big)$ is reduced, equivalently, if there is no asymptotically invariant sequence of unit vectors in $\ell^2(\Gamma/\Lambda)$. Recall also that $e(X)$ equals $e(\Gamma/\Lambda)$, where the latter is the number of ends of the quotient by Λ of any Cayley graph of Γ

relative to a finite generating set. The following result says that if $e(\Gamma/\Lambda) \geq 3$ and Γ/Λ is stable, then Λ comes essentially from a Riemann surface situation. More precisely:

THEOREM 1.6 ([DG05]). — *Let $\Gamma = \pi_1(M)$ be a Kähler group and $\Lambda < \Gamma$ a subgroup such that the Γ-space Γ/Λ is stable and $e(\Gamma/\Lambda) \geq 3$. Then there are a finite covering $M' \to M$ and a holomorphic map with connected fibers $h : M' \to S$ to a compact Riemann surface of genus $g \geq 2$ such that $\operatorname{Ker} h_* \subset \Lambda \cap \pi_1(M')$.*

Remark 1.7. — Napier and Ramachandran recently showed that, without the stability condition, the covering X associated to Λ admits a proper holomorphic mapping onto a Riemann surface, [NR08].

The number of relative ends $e(\Gamma/\Lambda)$ introduced by Houghton had been studied by Scott [Sco78] in the context of obtaining a relative version of Stallings' theorem. It was then realized by Sageev [Sag95] that the proper context for this problem is the one of group actions on CAT(0) cubical complexes; he showed that for a finitely generated group Γ there is $\Lambda < \Gamma$ with $e(\Gamma/\Lambda) \geq 2$ if and only if Γ admits an essential action on a CAT(0) cubical complex. Since this result, the question of cubing natural classes of groups has become a center of attention for geometric group theorists. In particular, right angled Artin groups and groups satisfying certain specific small cancellation properties have been shown to act properly and cocompactly on finite dimensional CAT(0) cubical complexes (see Example 5 in § 2).

When Γ is word hyperbolic, the condition on the number of ends in Theorem 1.6 can be somehow relaxed, but then a geometric condition has to be imposed on Λ.

COROLLARY 1.8. — *Assume that Γ is Kähler, word hyperbolic and that $\Lambda < \Gamma$ is quasiconvex with $e(\Gamma/\Lambda) \geq 2$. Then Γ is commensurable to Γ_g for some $g \geq 2$.*

Remark 1.9. — The case in which $e(\Gamma/\Lambda) = 2$ can be reduced to the situation of Theorem 1.6 after a rather involved argument which fully exploits hyperbolicity (see 5.5 and 6.5 in [DG05]).

This corollary has striking consequences in complex hyperbolic geometry. Let $\mathbb{H}^n_{\mathbf{C}}$ be the complex hyperbolic n-space and $\Gamma < \operatorname{Aut}(\mathbb{H}^n_{\mathbf{C}})$ a cocompact lattice; the quasiconvexity assumption on a subgroup $\Lambda < \Gamma$ means that the quotient by Λ of the closed convex hull in $\mathbb{H}^n_{\mathbf{C}}$ of the limit set $\mathcal{L}(\Lambda) \subset \partial\mathbb{H}^n_{\mathbf{C}}$ of Λ is compact, that is, Λ is convex cocompact.

COROLLARY 1.10. — *Let $\Gamma < \operatorname{Aut}(\mathbb{H}^n_{\mathbf{C}})$ be a cocompact lattice and assume that $n \geq 2$.*

1. *If $\Lambda < \Gamma$ is convex cocompact, then $\partial\mathbb{H}^n_{\mathbf{C}} \setminus \mathcal{L}(\Lambda)$ is connected.*
2. *The space $\Gamma\backslash\mathbb{H}^n_{\mathbf{C}}$ does not have the homotopy type of a locally* CAT(0) *cubical complex.*

The second assertion follows from Corollary 1.8 and a result of Sageev [Sag95] saying that if Γ acts on a finite dimensional CAT(0) cubical complex with an unbounded orbit, then there is a hyperplane $\mathcal{I} \subset X$ with $e\big(\Gamma/\operatorname{Stab}_\Gamma(\mathcal{I})\big) \geq 2$.

2. EXAMPLES

The following is a list of examples of Kähler groups. All of them are actually fundamental groups of smooth projective varieties.

1. Finite groups are Kähler, [Ser58].
2. Let $\mathcal{H}_{2k+1}$ be the real Heisenberg group of dimension $2k+1$; this group can be seen as the central extension of a $2k$-dimensional real vector space V by $\mathbf{R}$ where the cocycle is given by a symplectic form. Then a lattice $\Gamma < \mathcal{H}_{2k+1}$ is Kähler if and only if $k \geq 4$ ([Cam95], [ABC+96, Ch. 8, 4.1]).
3. The group given by the presentation
$$\Gamma_g = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g : \prod_{i=1}^g [\alpha_i, \beta_i] = e \right\rangle$$
is Kähler. It is the fundamental group of a smooth projective curve of genus g. These groups are ubiquitous in the theory of Kähler groups as they appear often in factorization theorems (see e. g. Corollary 1.3) and are usually referred to as surface groups. Incidentally, let $\overline{\Gamma}_g$ be the central extension by $\mathbf{Z}$ generating $\mathrm{H}^2(\Gamma_g, \mathbf{Z})$, that is
$$\overline{\Gamma}_g = \left\langle \overline{\alpha}_1, \ldots, \overline{\alpha}_g, \overline{\beta}_1, \ldots, \overline{\beta}_g, z : \prod_{i=1}^g [\overline{\alpha}_i, \overline{\beta}_i] = z,\ z \text{ is central} \right\rangle .$$
Then the cup product map $\Lambda \mathrm{H}^1(\overline{\Gamma}_g, \mathbf{R}) \to \mathrm{H}^2(\overline{\Gamma}_g, \mathbf{R})$ is the zero map, while $b_1(\overline{\Gamma}_g) > 0$ and hence $\overline{\Gamma}_g$ is not Kähler by § 1.2; observe that in this example $b_2(\overline{\Gamma}_g) > 0$.
4. A Kähler group is the fundamental group of a real compact 3-manifold if and only if it is finite and hence a finite subgroup of $O(4)$, [DS09].
5. If a Kähler group is a $C'(\frac{1}{6})$-small cancellation group then it is commensurable to Γ_g ([DG05], [Wis04]).
6. Let $\Gamma \times X \to X$ be a properly discontinuous action by automorphisms of a Kähler manifold X such that $\Gamma \backslash X$ is compact. If there is $\Gamma' < \Gamma$ of finite index acting freely on X then Γ is Kähler; this observation is due to J. Kollár.
7. The class of Kähler groups is closed under taking finite products and passing to subgroups of finite index.

8. Let G be a semisimple connected Lie group without compact factors and with finite center. Assume that the associated symmetric space X has a G-invariant complex structure; combining the Riemannian metric with the complex structure gives a G-invariant two-form on X (see (1)) which is therefore closed. Thus X is Kähler. If now G is linear and $\Gamma < G$ is a cocompact lattice, it follows from Selberg's lemma and Example 6 that Γ is Kähler. The fact that if Γ is torsion free and cocompact, then $\Gamma\backslash X$ is biholomorphic to a projective manifold, is a theorem. This leads to the following natural questions:
   (A) What about the case when $\Gamma\backslash X$ is not compact, but just has finite volume? Then, unless X is the Siegel upper half space of genus 1, 2 or the complex two-ball, Γ is Kähler (see [Tol90] for more details also on these exceptional cases).
   (B) What about the case when X is not Hermitian? Then it is conjectured that Γ is not Kähler. This is now established in many cases for instance if Γ is cocompact and G is almost simple of rank at least 20 (see [Klib] and [ABC$^+$96]).
   (C) What about the case when G is not linear? This leads in very specific cases to examples of Kähler groups which are not residually finite (see [ABC$^+$96, Ch. 8] and [Tol93]).

3. THE BIERI–NEUMANN–STREBEL INVARIANT OF A KÄHLER GROUP

In this section we will illustrate the interplay between geometric group theory and Kähler geometry by explaining some aspects of the description of the Bieri–Neumann–Strebel invariant – hereafter called the BNS invariant – of Kähler groups.

3.1. THE BNS INVARIANT [(2)] $\Sigma^1(\Gamma)$

Let Γ be a group, T a set endowed with a fixed point free involution $x \mapsto \overline{x}$ and a map $\ell : T \to \Gamma$ with $\ell(\overline{x}) = \ell(x)^{-1}$. The associated Cayley graph $Ca(\Gamma, T)$ has Γ as its set of vertices and $E = \{(g, x) : g \in \Gamma, x \in T\}$ as its set of edges with origin and terminus maps $o, t : E \to \Gamma$ given by $o(g, x) = g$, $t(g, x) = g\ell(x)$; see [Ser77] for conventions concerning this notion. We say, by abuse of language, that T is generating if $\ell(T)$ generates Γ. Given a homomorphism $\chi : \Gamma \to \mathbf{R}$, hereafter called character, let $Ca(\Gamma, T)_\chi$ denote the subgraph of $Ca(\Gamma, T)$ whose set of vertices is the submonoid $\Gamma_\chi = \{g \in \Gamma : \chi(g) \geq 0\}$ and whose set of edges is $\{(g, x) \in E : \chi(g) \geq 0, \chi\big(g\ell(x)\big) \geq 0\}$. The following fact is then the starting point of the theory of BNS-invariants:

Proposition 3.1 ([BS, Theorem 2.1]). — *Let T, T' be finite generating sets of Γ and $\chi : \Gamma \to \mathbf{R}$ a homomorphism. Then $Ca(\Gamma, T)_\chi$ is connected if and only if $Ca(\Gamma, T')_\chi$ is.*

2. For a comprehensive treatment, see [BNS87] or [BS].

For a finitely generated group Γ we say that $\chi \in \mathrm{Hom}(\Gamma, \mathbf{R}) \setminus \{0\}$ is regular if $Ca(\Gamma, T)_\chi$ is connected and exceptional otherwise. Let $S(\Gamma)$ denote the sphere consisting of all half-rays in $\mathrm{Hom}(\Gamma, \mathbf{R}) \setminus \{0\}$; the BNS invariant of Γ is the subset $\Sigma^1(\Gamma) \subset S(\Gamma)$ represented by regular characters, that is:

Definition 3.2. — *$\Sigma^1(\Gamma) := \big\{[\chi] = \mathbf{R}_{>0}\chi : \chi \in \mathrm{Hom}(\Gamma, \mathbf{R}) \setminus \{0\}, Ca(\Gamma, T)_\chi$ is connected$\big\}$ and $E^1(\Gamma)$ denotes the complement of $\Sigma^1(\Gamma)$ in $S(\Gamma)$.*

The following examples are obtained by direct computation:

Examples. — 1. For a finitely generated abelian group A, $\Sigma^1(A) = S(A)$.

2. For the free group $\mathbb{F}_n$ on $n \geq 2$ generators, $\Sigma^1(\mathbb{F}_n) = \emptyset$.

3. For the solvable group $\Gamma = \left\{ \begin{pmatrix} 2^n & x \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z}, x \in \mathbf{Z}[\tfrac{1}{2}] \right\}$, $\Sigma^1(\Gamma) = \big\{[-\chi]\big\}$, where $\chi \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 1$, whereas $S(\Gamma) = \big\{[\chi], [-\chi]\big\}$.

A fundamental aspect of the theory for a finitely generated group Γ is the connection between the finiteness properties of the kernel N of a surjective group homomorphism $\pi : \Gamma \to Q$ and the BNS invariants of Γ and Q. We start with the following simple observations. A finite generating set T for Γ, and hence for Q, leads to a Galois covering $\pi_* : Ca(\Gamma, T) \to Ca(Q, T)$ with Galois group N and, for any non-zero character $\chi : Q \to \mathbf{R}$, to a Galois covering

$$\pi_{*,\chi} : Ca(\Gamma, T)_{\chi \circ \pi} \to Ca(Q, T)_\chi \,. \tag{2}$$

This implies at once that if $S(\Gamma, N) \subset S(\Gamma)$ denotes the great sphere cut out by the non-zero characters of Γ vanishing on N, then $\pi^*\big(E^1(Q)\big) \subset E^1(\Gamma) \cap S(\Gamma, N)$. If now N is finitely generated, in which case one can choose T finite so that $\ell(T)$ contains a generating set of N, the covering map (2) has connected fibers, and thus

$$S(\Gamma, N) \cap \Sigma^1(\Gamma) = \pi^*\big(\Sigma^1(Q)\big) \,.$$

If in addition Q is abelian we deduce, using Example 1.1, that $S(\Gamma, N) \subset \Sigma^1(\Gamma)$. We have then the fundamental

Theorem 3.3 ([BS, Theorem 4.2]). — *Let $1 \to N \to \Gamma \to Q \to 1$ be an exact sequence where Γ is finitely generated and Q is abelian. The group N is finitely generated if and only if $S(\Gamma, N) \subset \Sigma^1(\Gamma)$. In particular the commutator subgroup $[\Gamma, \Gamma]$ is finitely generated if and only if $\Sigma^1(\Gamma) = S(\Gamma)$.*

An important point in the proof is a characterization of $\Sigma^1(\Gamma)$ in terms of a "locally finite set of inequalities" implying in particular that $\Sigma^1(\Gamma)$ is open. Thus:

Proposition 3.4. — *$E^1(\Gamma) \subset S(\Gamma)$ is a closed $\mathrm{Aut}(\Gamma)$-invariant subset.*

Recall that a group Q is metabelian if its first commutator subgroup $[Q, Q]$ is abelian; Example 3 above is metabelian. The invariant Σ^1 is particularly relevant for the study of metabelian groups: for example a metabelian group Q is finitely presented if and only if $S(Q) = \Sigma^1(Q) \cup (-\Sigma^1(Q))$ (cf. [BS80]). Also in our context metabelian groups will play an essential role. For the moment we wish to have the following application of the results stated so far:

PROPOSITION 3.5. — *Let Q_g be the largest metabelian quotient of the surface group Γ_g and assume that $g \geq 2$. Then $E^1(Q_g) = S(Q_g)$.*

The largest metabelian quotient Q of a group Γ is the quotient of Γ by $\mathcal{D}^{(2)}\Gamma$, where $\mathcal{D}\Gamma := [\Gamma, \Gamma]$ and $\mathcal{D}^{(2)}\Gamma = [\mathcal{D}\Gamma, \mathcal{D}\Gamma]$.

Proof. — The group $\mathcal{D}\Gamma_g$ is not finitely generated since it is the first homology group of a connected surface which is an infinite, non-simply connected Galois covering of a compact surface. By means of a symplectic basis, indentify $S(Q_g)$ with $\mathbf{P}(\mathbf{R}^{2g})$; then every element in $Sp(2g, \mathbf{Z})$ lifts to $\mathrm{Aut}\, Q_g$, since it even lifts to $\mathrm{Aut}\, \Gamma_g$. Using that every $Sp(2g, \mathbf{Z})$-orbit in $\mathbf{P}(\mathbf{R}^{2g})$ is dense and that $E^1(Q_g)$ is closed (Proposition 3.4) and not empty, one concludes that $E^1(Q_g) = S(Q_g)$. □

3.2. A FINITENESS THEOREM OF BEAUVILLE

An important ingredient in the study of solvable quotients of Kähler groups is the following structure theorem of their metabelian quotients, in a situation which is quite opposite to the one of surface groups.

COROLLARY 3.6 ([Bea92]). — *Let $Q = \Gamma/\mathcal{D}^{(2)}\Gamma$ be the largest metabelian quotient of a Kähler group Γ. Assume that $E^1(Q) = \emptyset$. Then Q is virtually nilpotent.*

Recall that the condition $E^1(Q) = \emptyset$ is equivalent to the condition that $\mathcal{D}Q$ is finitely generated; under this condition the solvable group Q acts linearly in the finite dimensional space $\mathcal{D}Q \otimes \mathbf{C}$ and it is therefore clear that the set

$$\mathcal{E}^1(\Gamma, \mathbf{C}^\times) = \left\{\rho \in \mathrm{Hom}(\Gamma, \mathbf{C}^\times) : \mathrm{H}^1(\Gamma, \mathbf{C}_\rho) \neq 0\right\}$$

has to play an important role. This set is the Green–Lazarsfeld set of Γ and has been the topic of numerous investigations (see [Ara97], [Bea92], [Cam01], [GL87], [GL91], [PR04], [Sim93b]) culminating in [Del08], where the precise structure of $\mathcal{E}^1(\Gamma, K^\times)$ – where K is an arbitrary field – is described; it relies on the description of the BNS invariant of Γ.

We have:

PROPOSITION 3.7 ([Bea92]). — *Let Γ be a Kähler group and $Q = \Gamma/\mathcal{D}^{(2)}\Gamma$ its largest metabelian quotient. Assume that $\mathcal{D}Q$ is finitely generated. Then $\mathcal{E}^1(\Gamma, \mathbf{C}^\times)$ is finite and consists of torsion characters.*

Proof. Consider the restriction map

$$\bigoplus_{1\neq\rho\in\mathcal{E}^1} \mathrm{H}^1(Q,\mathbf{C}_\rho) \to \mathrm{Hom}(\mathcal{D}Q,\mathbf{C})$$

obtained by restricting cocycles to $\mathcal{D}Q$. It is a linear algebra exercise involving the Vandermonde determinant, that this map is injective. In particular if $\mathcal{D}Q$ is finitely generated, then $\mathcal{E}^1(Q,\mathbf{C}^\times)$ is finite. Since elements in $\mathcal{E}^1(\Gamma,\mathbf{C}^\times)$ correspond to homomorphisms of Γ into the affine group of $\mathbf{C}$ which is metabelian, then $\mathcal{E}^1(\Gamma,\mathbf{C}^\times) = \mathcal{E}^1(Q,\mathbf{C}^\times)$. By Corollary 3.6 of [Bea92], we know that isolated points in $\mathcal{E}^1(\Gamma,\mathbf{C}^\times)$ are unitary, thus $\mathcal{E}^1(\Gamma,\mathbf{C}^\times)$ consists of a finite number of unitary homomorphisms; since $\mathrm{Aut}(\mathbf{C})$ acts by postcomposition on $\mathcal{E}^1(\Gamma,\mathbf{C}^\times)$, a theorem of Kronecker then implies the proposition. □

Proof of Corollary 3.6. The subgroup $Q^\circ = \bigcap_{\chi\in\mathcal{E}^1} Ker\chi$ is of finite index in Q and its action on $\mathcal{D}Q\otimes\mathbf{C}$ is by unipotent endomorphisms. □

3.3. THE FACTORIZATION THEOREM

Here we will discuss the central result which is a description of the set of exceptional characters of the fundamental group $\Gamma = \pi_1(X)$ of a compact Kähler manifold. This description is in terms of factorizations of X over certain Riemann surfaces, but leads ultimately to a purely group theoretical description of $E^1(\Gamma)$.

We use the notation $\chi \mapsto \omega_\chi$ for the canonical isomorphism between $\mathrm{Hom}(\Gamma,\mathbf{R})$ and the space $\mathcal{H}^1(X)$ of harmonic real 1-forms on X.

Theorem 3.8 ([Del10]). — *The homomorphism $\chi:\Gamma\to\mathbf{R}$ is exceptional if and only if there are a holomorphic map with connected fibers $f: X\to S^{orb}$ onto a hyperbolic orbi-Riemann surface S^{orb} of genus $g\geq 1$ and a closed holomorphic 1-form η on S^{orb} with $\omega_\chi = \Re(f^*\eta)$.*

In the case $\chi(\Gamma)=\mathbf{Z}$, the result is due to Napier and Ramachandran (see [NR01, Thm 4.3]).

The object S^{orb} consists of an underlying Riemann surface S and finitely many marked points $p_1,\dots,p_n$ in S each having an integer "multiplicity" $m_i\geq 2$; then S^{orb} is hyperbolic if

$$\chi(S^{orb}) := (2-2g) - \sum_{i=1}^n \left(1-\frac{1}{m_i}\right) < 0\,.$$

In this case S^{orb} can be uniformized by the Poincaré disk $\mathbb{D}$ and occurs as the quotient of $\mathbb{D}$ by a faithful proper action of the orbifold fundamental group

$$\pi_1(S^{orb}) = \left\langle \alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g,f_1,\dots,f_n : \prod_{i=1}^g[\alpha_i,\beta_i]\prod_{i=1}^n f_i = e,\ f_i^{m_i}=e\right\rangle.$$

The marked points form the set of critical values of f; in this context for f to be holomorphic means that if x is a critical point with $f(x)=p_i$, then locally f lifts to a holomorphic map into a cover with a branching point of multiplicity m_i above p_i. In this situation the map f in Theorem 3.8 induces a surjective homomorphism $f_* : \Gamma \to \pi_1(S^{orb})$.

For the "easy" implication in Theorem 3.8, observe that if $\chi(S^{orb}) < 0$ then every (nonzero) character of $\pi_1(S^{orb})$ is exceptional. If S^{orb} is a genuine surface of genus $g \geq 2$, this follows from Proposition 3.5; the general case can be reduced to the previous one by passing to an appropriate subgroup Γ of finite index in $\pi_1(S^{orb})$ and using that a character χ of $\pi_1(S^{orb})$ is regular if and only if $\chi|_\Gamma$ is as well ([BNS87, Prop. 3.2(ii)]).

We now indicate the strategy for the proof of the factorization statement in Theorem 3.8, which relies on the following theorem of Simpson.

Theorem 3.9 ([Sim93a]). — *Let X be a compact Kähler manifold, θ a closed holomorphic 1-form on X which is non-zero, $p : Y \to X$ a covering such that $p^*(\theta)$ is exact, and let $g : Y \to \mathbf{R}$ be a primitive of $\Re(p^*\theta)$. Then either*

1. *the fiber $g^{-1}(v)$ is connected and $\pi_1\big(g^{-1}(v)\big) \to \pi_1(Y)$ surjects for all $v \in \mathbf{R}$,*

or

2. *there are a hyperbolic orbi-Riemann surface S^{orb}, a holomorphic map $f : X \to S^{orb}$ with connected fibers and a closed holomorphic 1-form η on S with $f^*(\eta) = \theta$.*

With this at hand we can sketch a proof of Theorem 3.8. Let $\chi : \Gamma \to \mathbf{R}$ be an exceptional character, $p : Y \to X$ the maximal abelian cover of X and $g : Y \to X$ a primitive of $p^*(\omega_\chi)$. Then

$$g(\gamma y) = \chi^{ab}(\gamma) + g(y) \tag{3}$$

for all $y \in Y$ and $\gamma \in \Gamma^{ab} := \Gamma/\mathcal{D}\Gamma$, where $\chi^{ab} \in \mathrm{Hom}(\Gamma^{ab}, \mathbf{R})$ corresponds to χ. Since Γ^{ab} is finitely generated abelian, χ^{ab} is regular (see § 3.1) and thus $Ca(\Gamma^{ab}, T)_{\chi^{ab}}$ is connected; here T is some finite generating set of Γ. From this and the equivariance property (3) one deduces easily that the set $g^{-1}\big([0,\infty]\big)$ has a unique connected component, say Y_0, on which g is unbounded. Consider now $g \circ \pi$, where $\pi : \widetilde{X} \to Y$ is the universal covering projection. Clearly any connected component of $(g \circ \pi)^{-1}\big([0,\infty]$ on which $g \circ \pi$ is unbounded is a connected component of $\pi^{-1}(Y_0)$; since X is exceptional, $Ca(\Gamma, T)_\chi$ is not connected and hence there are several connected components of $(g \circ \pi)^{-1}\big([0,\infty]\big)$ on which $g \circ \pi$ is unbounded, which, by the preceding remark, implies that $\pi^{-1}(Y_0)$ is not connected. This implies that the morphism $\pi_1(Y_0) \to \pi_1(Y)$ is not surjective. Now pick $y \in Y_0$ and $v = g(y)$. Then either $g^{-1}(v) \subset Y$ is not connected, or $g^{-1}(v) \subset Y_0$ and thus $\pi_1\big(g^{-1}(v)\big) \to \pi_1(Y)$ is not surjective. At any rate, it is the second alternative of Simpson's theorem which applies.

3.4. SOLVABLE GROUPS AND METABELIAN QUOTIENTS

What allows one to get applications of the factorization theorem (Theorem 3.8) is a very efficient way to detect solvable groups which are not virtually nilpotent. Such groups must have special quotients. We will in the sequel say that a group R is just not virtually nilpotent if every proper quotient of R is virtually nilpotent but R itself is not. We observe the following:

Proposition 3.10. — *Every finitely generated group which is not virtually nilpotent admits a quotient which is just not virtually nilpotent.*

This follows easily from the fact that finitely generated nilpotent groups are finitely presented and Zorn's lemma. The following result is based essentially on arguments of Groves [Gro78] and a proof can be found in [Bre07]. Here we present a sketch, with a few simplifications.

Theorem 3.11. — *Let Q be finitely generated, solvable and just not virtually nilpotent. Then Q is virtually metabelian.*

If Q is such a group, it has the Noetherian property: every ascending chain $N_1 \subset N_2 \subset \dots$ of normal subgroups stabilizes: indeed, if $N_j \neq e$, G/N_j is finitely generated nilpotent and has this property. In particular the Fitting subgroup $Fit(Q)$, which is the subgroup of Q generated by all normal nilpotent subgroups is nilpotent as well. Then in the situation of Theorem 3.11 we have

Lemma 3.12. — *The subgroup $Fit(Q)$ is abelian, and it is either torsion-free or p-torsion for some prime p.*

Proof. (1) If $N \lhd Q$ is a normal nilpotent subgroup with $\mathcal{D}N \neq e$, let $H < Q$ be of finite index with $H/\mathcal{D}N$ nilpotent. Clearly $HN/\mathcal{D}N \subset G/\mathcal{D}N$ is nilpotent and $N < HN$ is nilpotent, so that by Hall's criterion, [Rob96, 5.2.10], HN is nilpotent; this is a contradiction and hence $\mathcal{D}N = (e)$.

(2) Any pair N_1, N_2 of nontrivial normal subgroups must intersect since $Q/N_1 \cap N_2$ is a subgroup of the virtually nilpotent group $Q/N_1 \times Q/N_2$. Thus if $Fit(Q)$ is not torsion free, it can only have p-torsion for a unique prime. □

Without loss of generality, assume that $L := Q/Fit(Q)$ is nilpotent (instead of virtually nilpotent). Let $A = \mathbf{Z}$ or $\mathbf{F}_p[T]$; if $Fit(Q)$ is p-torsion we make it into an A-module by letting T act via conjugation of some fixed central element of infinite order in L. At any rate one verifies that $Fit(Q)$ is a torsion free A-module and a finitely generated $A[L]$-module; the latter follows again from the Noetherian property. By applying a theorem of Hall (see [Rob96, 15.4.3]), one shows:

Proposition 3.13. — *If $K = \mathbf{Q}$ or $\mathbf{F}_p(T)$, $Fit(G) \otimes_A K$ is finite dimensional over K.*

Let $\rho : Q \to \mathrm{GL}\big(Fit(Q) \otimes_A K\big)$ be the resulting n-dimensional representation. Then, since $\rho(Q) = \rho(L)$ is linear nilpotent, $\rho(\mathcal{D}Q)$ acts unipotently and thus for every $v \in Fit(Q)$ and $g_1, \dots, g_n \in \mathcal{D}Q$, one has that $\big[g_1, \dots, [g_n, v]\big] = e$. Thus $\mathcal{D}Q$ is nilpotent and hence in $Fit(Q)$.

3.5. SOLVABLE QUOTIENTS AND FACTORIZATION

We are now in a position to deduce:

Corollary 3.14 ([Del10]). — *Let $\Gamma = \pi_1(X)$ be the fundamental group of a compact Kähler manifold. Then either*

(1) *Any solvable quotient of Γ is virtually nilpotent*

or

(2) *there is a subgroup $\Gamma' < \Gamma$ of finite index and a surjection $\Gamma' \to \Gamma_g$ onto a surface group of genus $g \geq 2$.*

Remark 3.15. — This result had been obtained previously in the case of solvable linear quotients by Campana [Cam01], following previous work by Arapura and Nori for fundamental groups of projective varieties [AN99]; see also [Kat97] and [Bru03].

Observe that we can deduce now from Corollary 3.14 the result announced in the introduction, namely:

Corollary 3.16 ([Del10]). — *A Kähler group which is solvable is virtually nilpotent.*

Proof. Assume that S is a quotient of Γ which is solvable but not virtually nilpotent. Let R be a just non virtually nilpotent quotient (see Proposition 3.10). Since R is solvable, let R' be a metabelian subgroup of finite index (Theorem 3.11) and Γ' its inverse image in Γ which is of finite index again and hence Kähler. Then R' is a quotient of $Q' := \Gamma'/\mathcal{D}^{(2)}(\Gamma')$ and Q' cannot be virtually nilpotent, which by Corollary 3.6 implies that $\mathcal{D}Q'$ and hence $\mathcal{D}\Gamma'$ is not finitely generated; but then Theorem 3.3 implies that $E^1(\Gamma') \neq \emptyset$ and the factorization theorem (Theorem 3.8) applies. $\square$

Acknowledgments. I thank Thomas Delzant for extensive explanations, Ralph Strebel for detailed comments, as well as Paul Biran, Alessandra Iozzi, Dieter Kotschick and Doming Toledo for helpful suggestions.

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