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THE ACC CONJECTURE FOR LOG CANONICAL THRESHOLDS [after de Fernex, Ein, Mustaţă, Kollár]

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INTRODUCTION

The minimal model conjecture asserts that every algebraic variety X not covered by rational curves is birational to a projective variety Y which is negatively curved in the sense that the first Chern class $c_1(Y)$ has degree ≤ 0 on all curves in Y, called a minimal model. The problem is fundamental to all kinds of classification problems in algebraic geometry. The minimal model conjecture is known for all varieties of general type [BCHM] and for all varieties of dimension at most 4, but the full conjecture remains wide open. (Although the minimal model conjecture should be true in any characteristic, the results mentioned work over a field of characteristic zero. For these problems, we lose nothing by working over the complex numbers. Birkar [Birkarexist] summarizes the known results on the minimal model conjecture in dimensions up to 5.)

Shokurov showed that the minimal model conjecture would follow from the ascending chain condition for a certain invariant of singularities, the minimal log discrepancy (mld), together with a semicontinuity property for minimal log discrepancies [Shokurovmld]. The minimal log discrepancy is a rational number associated to a given singularity, with bigger numbers corresponding to milder singularities. Shokurov's ACC conjecture says that the set of minimal log discrepancies of all singularities of a given dimension is a subset of the real line that contains no infinite increasing sequence. Let us say vaguely why this would imply the minimal model conjecture. Starting from any projective variety, after contracting finitely many divisors, we can make it closer and closer to a minimal model by a sequence of birational maps called flips. We get a minimal model if we can show that no infinite sequence of flips is possible. But each flip improves the singularities as measured by minimal log discrepancies, and so the ACC conjecture would imply that the sequence of flips terminates.

Shokurov also conjectured the ACC property for another invariant of singularities, the log canonical threshold (lct). This conjecture should be a natural preliminary to the ACC conjecture for minimal log discrepancies, since log canonical thresholds are simpler than minimal log discrepancies and have been studied for a long time in singularity theory. In particular, ACC for log canonical thresholds is known in dimension at most 3 while ACC for minimal log discrepancies is known only in dimension at most 2, by Alexeev [Alexeev1, Alexeev2]; moreover, ACC for minimal log discrepancies in dimension 3 would imply ACC for log canonical thresholds in dimension 4 [SB, Corollary 1.10]. Log canonical thresholds have an elementary analytic definition: for an analytic function f on $\mathbf{C}^n$, the log canonical threshold of the hypersurface $f = 0$ at a point p is the supremum of the real numbers s such that $|f|^{-s}$ is L^2 near p. Estimates of log canonical thresholds have a variety of applications in algebraic geometry, including the construction of Kähler-Einstein metrics on many Fano varieties [DK] and the proof of non-rationality for many Fano varieties [DEMrigid].

The ACC conjecture for log canonical thresholds has some implication for the minimal model conjecture, albeit a limited one. By Birkar, the minimal model conjecture in dimension $n-1$ (for pairs $(X; B)$ with $K_X + B$ pseudo-effective) implies the minimal model conjecture for pairs $(X; B)$ of dimension n *with* $K_X + B$ *effective* [Birkarexist2]. If we also know ACC for log canonical thresholds on singular varieties of dimension n, then we can deduce termination of flips for pairs $(X; B)$ of dimension n with $K_X + B$ effective [BirkarACC]. Termination is a stronger statement than existence of minimal models (because it says that any sequence of flips will lead to a minimal model).

A recent advance is the proof of ACC for log canonical thresholds on smooth varieties of any dimension by de Fernex, Ein, and Mustaţă [DEM]. Their method covers many singular varieties as well, including quotient singularities and local complete intersections. In dimension 3, every terminal singularity is a quotient of a hypersurface singularity by a finite group, and so de Fernex-Ein-Mustaţă's methods reprove the general ACC conjecture for log canonical thresholds in dimension 3. In dimensions at least 4, there seems to be no hope of a comparably explicit description of terminal singularities. Nonetheless, de Fernex-Ein-Mustaţă's work provides striking new evidence for the ACC conjectures, suggesting that they form a plausible approach toward the minimal model conjecture. The final version of de Fernex-Ein-Mustaţă's argument, incorporating contributions by Kollár, is short and simple.

This exposition owes a lot to Kollár's excellent survey of the ACC conjecture for log canonical thresholds [KACC]. Thanks to Ofer Gabber for suggesting Corollary 1.7.

1. INTRODUCTION TO LOG CANONICAL THRESHOLDS

Definition 1.1. — *Let f be a holomorphic function in a neighborhood of a point $p \in \mathbf{C}^n$. The log canonical threshold of f at p is the number $c = \mathrm{lct}_p(f)$ such that*

— *$|f|^{-s}$ is L^2 in a neighborhood of p for $s < c$, and*

— *$|f|^{-s}$ is not L^2 in a neighborhood of p for $s > c$.*

Thus $\mathrm{lct}_p(f) = \infty$ if $f(p) \neq 0$, and by convention $\mathrm{lct}_p(0) = 0$.

The log canonical threshold is a natural measure of the complexity of the zero set of f near p. It was considered by Atiyah [Atiyah] and Bernstein [Bernstein], and further explored by Arnold, Gusein-Zade and Varchenko [AGZV, v. 2, Section 13.1.5] and Kollár [Ksing]. The main textbook treatment of log canonical thresholds is in Lazarsfeld's book [Lazarsfeld, Section 9.3].

Shokurov had the idea that the set of all possible values of log canonical thresholds in a given dimension should have special properties [Shokurovprob].

DEFINITION 1.2. — *Let $\mathcal{HT}_n$ be the set of log canonical thresholds of all possible holomorphic functions of n variables vanishing at* 0. *That is,*

$$\mathcal{HT}_n = \{\mathrm{lct}_0(f) : f \in O_{0,\mathbf{C}^n}, f(0) = 0\} \subset \mathbf{R}.$$

The name $\mathcal{HT}_n$ indicates that these are "hypersurface thresholds". Indeed, it is easy to see that $\mathrm{lct}_0(gf) = \mathrm{lct}_0(f)$ for $g(0) \neq 0$, which says that $\mathrm{lct}_0(f)$ only depends on the hypersurface $\{f = 0\}$ near $0 \in \mathbf{C}^n$ (considered with multiplicities). We get the same set $\mathcal{HT}_n$ if we allow f to run through all polynomials or all formal power series over any algebraically closed field of characteristic zero, using the algebraic definition of the log canonical threshold in Section 3 [KACC, Section 5].

The function $|z|^{-s}$ is L^2 on a neighborhood of the origin if and only if $s < 1$. It follows that, for a holomorphic function f of one variable,

$$\mathrm{lct}_p(f) = \frac{1}{\mathrm{mult}_p(f)}.$$

As a result,

$$\mathcal{HT}_1 = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\right\}.$$

There is also a complete description of the set $\mathcal{HT}_2$, by Varchenko [Varchenko], [KSC, Theorem 6.40], [KACC, Equation (15.5)]:

$$\mathcal{HT}_2 = \left\{\frac{c_1 + c_2}{c_1c_2 + a_1c_2 + a_2c_1} : a_1, a_2, c_1, c_2 \in \mathbf{N}, c_1 + c_2 \geq 1, \right.$$
$$\left. a_1 + c_1 \geq \max\{2, a_2\}, a_2 + c_2 \geq \max\{2, a_1\}\right\} \cup \{0\}.$$

The sets $\mathcal{HT}_n$ are not known for $n \geq 3$, and it may be unreasonable to expect an explicit description. Nonetheless, they have remarkable properties. First, Atiyah used resolution of singularities to prove:

LEMMA 1.3. — *All log canonical thresholds are rational and lie between* 0 *and* 1. *That is,* $\mathcal{HT}_n \subset \mathbf{Q} \cap [0, 1]$.

We will discuss de Fernex, Ein, and Mustaţă's theorem:

THEOREM 1.4 (ACC conjecture, smooth case). — *For any n, there is no infinite increasing sequence in $\mathcal{HT}_n$.*

By contrast, every rational number between 0 and 1 is the log canonical threshold of a holomorphic function in some number of variables. Also, there are many decreasing sequences of log canonical thresholds in a given dimension, as we see from the example [AGZV, v. 2, Section 13.3.5], [Ksing, Proposition 8.21]:

Lemma 1.5. —

$$\mathrm{lct}_0(z_1^{a_1} + \cdots + z_n^{a_n}) = \min\left\{1, \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right\}.$$

Kollár described all limits of decreasing sequences of log canonical thresholds on smooth varieties of dimension n:

Theorem 1.6 (Accumulation conjecture, smooth case). — *The set of accumulation points of $\mathcal{HT}_n$ is $\mathcal{HT}_{n-1} - \{1\}$.*

In particular, $\mathcal{HT}_n$ is a closed subset of the unit interval, although it is contained in the rational numbers.

The ACC theorem implies that there is some $\epsilon_n > 0$ such that no log canonical threshold on a smooth n-dimensional variety is in $(1 - \epsilon_n, 1)$ (the smooth case of the Gap conjecture). The proofs are nonconstructive, and so there is no explicit lower bound for ϵ_n in general. There is a conjecture for the optimal value of ϵ_n. Consider the sequence defined by $c_{n+1} = c_1 \cdots c_n + 1$ starting with $c_1 = 2$. It is called Euclid's or Sylvester's sequence, and starts as:

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \cdots$$

The definition of c_i implies that

$$\sum \frac{1}{c_i} = 1 - \frac{1}{c_{n+1} - 1} = 1 - \frac{1}{c_1 \cdots c_n}.$$

In particular, by Lemma 1.5,

$$\mathrm{lct}_0(z_1^{c_1} + \cdots + z_n^{c_n}) = 1 - \frac{1}{c_{n+1} - 1}.$$

Kollár conjectured that this is the extreme case [KACC]. That is, the conjecture is that no log canonical threshold in dimension n lies in the interval

$$\left(1 - \frac{1}{c_{n+1} - 1}, 1\right).$$

This optimal Gap conjecture is known in dimension 3 ($\epsilon_3 = 1/42$) by Kollár [Ksing]. The set $\mathcal{HT}_3$ is not known, but Kuwata determined $\mathcal{HT}_3 \cap [5/6, 1]$ [Kuwata]. The ACC conjecture for functions on singular 3-folds, which we have not formulated yet, was proved by Alexeev [Alexeev1] and the Accumulation conjecture on singular 3-folds by McKernan-Prokhorov [MP]

Finally, we can rephrase the ACC theorem, Theorem 1.4, as saying that the set $\mathcal{HT}_n$ turned upside down is well-ordered. Gabber pointed out that the Accumulation theorem, Theorem 1.6, determines the order type of $\mathcal{HT}_n$ by a simple induction:

Corollary 1.7. — *For each positive integer n, the set $\mathcal{HT}_n$ turned upside down has order type* $\omega^n + 1$.

2. FORMULA FOR THE LOG CANONICAL THRESHOLD

In this section, we give Atiyah's formula for the log canonical threshold in terms of an embedded resolution of the hypersurface $\{f = 0\}$.

Lemma 2.1. — *If $f(p) \neq 0$, then* $\mathrm{lct}_p(f) = \infty$. *If $f(p) = 0$, then* $0 \leq \mathrm{lct}_p(f) \leq 1$.

Proof. The first claim is clear. So assume that $f(p) = 0$. As we said, for a one-variable holomorphic function $f(z)$, we have $\mathrm{lct}_p(f(z)) = 1/\mathrm{mult}_p(f)$. In several variables, pick a smooth point q near p on the hypersurface $\{f = 0\}$. We can choose local coordinates near q such that $f = (\mathrm{unit}) z_1^m$ for some m. If a function is L^2 on some neighborhood of p, then it is L^2 near some such point q, and so $\mathrm{lct}_p(f) \leq \mathrm{lct}_q(f) = 1/m$. □

In several variables, the *multiplicity* at 0 of a holomorphic function f means the smallest degree of a nonzero term in the power series expansion of f at 0. In any dimension, the log canonical threshold differs by a bounded factor from the multiplicity:

$$\frac{1}{\mathrm{mult}_p(f)} \leq \mathrm{lct}_p(f(z_1, \ldots, z_n)) \leq \frac{n}{\mathrm{mult}_p(f)}$$

[KACC, (11.5), (20.1)]. The ACC property is trivially true for the invariant $1/\mathrm{mult}_p(f)$ of functions f, as that only takes the values $\{1, 1/2, 1/3, \ldots\} \cup \{0\}$, but the ACC property for log canonical thresholds is a deeper problem.

Let $\omega = dz_1 \wedge \cdots \wedge dz_n$. Then $|f|^{-s}$ is locally L^2 if and only if on any compact set $K \subset U$, the integral

$$\int_K (f\overline{f})^{-s} \omega \wedge \overline{\omega}$$

is finite. Let $\pi : X \to U$ be a proper bimeromorphic morphism. We can rewrite this integral as

$$(*) \qquad \int_K (f\overline{f})^{-s} \omega \wedge \overline{\omega} = \int_{\pi^{-1}(K)} ((f \circ \pi)\overline{(f \circ \pi)})^{-s} \pi^*\omega \wedge \pi^*\overline{\omega}.$$

Since $X \to U$ is proper, it suffices to show that the integral on the right is finite near any given point of X.

By Hironaka, we can choose $\pi : X \to U$ to be an embedded resolution of the hypersurface $f = 0$. That means that X is smooth and the zero set of $f \circ \pi$ together with the exceptional set of f form a normal crossing divisor. That is, at any point $q \in X$, we can choose local coordinates $x_1, \ldots, x_n$ such that

$$f \circ \pi = (\mathrm{unit}) \prod_i x_i^{a_i(q)}$$

and

$$ ^* ! = (\text{unit}) \prod_i x_i^{e_i(q)} dx_1 \wedge \cdots \wedge dx_n; $$

where $a_i(q) = \text{mult}_{x_i=0}(f \circ \ )$ and $e_i(q) = \text{mult}_{x_i=0}\text{Jac}(\ )$. Here Jac denotes the complex Jacobian

$$ \text{Jac}(\ ) = \det\left(\frac{@z_i}{@x_j}\right): $$

Then the integral (?) is finite near a point $q \in X$ if and only if

$$ \int \prod_i (x_i \overline{x_i})^{e_i(q)-sa_i(q)} dV = (\text{const}) \prod_i \int (x_i\overline{x_i})^{e_i(q)-sa_i(q)} dx_i \wedge d\overline{x_i} $$

is finite. This holds if and only if $e_i(q) - sa_i(q) > -1$ for every i, that is, when $s < (e_i(q)+1)=a_i(q)$. This gives the formula for the log canonical threshold:

$$ \text{lct}_p(f) = \min\left\{\frac{1+\text{mult}_E \text{Jac}(\ )}{\text{mult}_E(f \circ \ )} : \text{ for those } E \text{ such that } p \in \ (E)\right\}: $$

It follows in particular that the log canonical threshold of any holomorphic function f is a rational number. We can rewrite the formula in the language of divisors. Let D be the divisor of a holomorphic function f, meaning the linear combination with integer coefficients of the irreducible components of the hypersurface $f = 0$, $D = \sum d_i D_i$ where d_i is the multiplicity of f along D_i. We can interpret the Jacobian of a resolution $: X \to U$ as describing the difference between the canonical divisors of X and U:

$$ K_X = \ ^*(K_U) + \sum_i e_i E_i: $$

Also, $\ ^*(D)$ means the divisor of the function $f \circ \ $, $\ ^*(D) = \sum a_i E_i$. In this notation,

$$ \text{lct}_0(D) = \min\left\{\frac{e_i+1}{a_i} : a_i > 0\right\}: $$

Equivalently, in the terminology of minimal model theory, the log canonical threshold of a divisor D on a variety X is the maximum real number c such that the pair $(X; cD)$ is log canonical [Lazarsfeld, Definition 9.3.9]. This definition makes sense on singular varieties X as well, and the ACC conjecture and related conjectures make sense in that generality.

3. THE LOG CANONICAL THRESHOLD OF AN IDEAL OF FUNCTIONS

The definition of the log canonical threshold of a function was generalized to ideals of functions by de Fernex, Ein, and Mustaţă [DEM]. They prove the ACC property in the greater generality of ideals (on a smooth variety). This stronger statement turns out to be easier to prove, because working with ideals makes it easier to concentrate on the behavior of a function at a single point.

Let $\mathfrak{a}$ be a sheaf of ideals on an excellent noetherian scheme X over $\mathbf{Q}$. (The main example in this paper is the spectrum of a power series ring over a field of characteristic zero.) By Hironaka, extended by Temkin [Hironaka, Kres, Temkin], there is a *log resolution* $Y \to X$ of the ideal $\mathfrak{a}$. That is, $Y \to X$ is a proper birational morphism, Y is regular, the union of the exceptional locus of $Y \to X$ with the inverse image of the zero set of $\mathfrak{a}$ is a divisor $\cup E_i$ with simple normal crossings, and the ideal $\mathfrak{a} \cdot O_Y$ is a principal ideal, of the form $\mathfrak{a} \cdot O_Y = O_Y(-\sum a_i E_i)$ for some natural numbers a_i. That is, $\mathfrak{a} \cdot O_Y$ is the ideal of regular functions on Y which vanish to order at least a_i on the divisor E_i for each i. The image in X of a divisor E_i is also called the *center* of E_i on X.

Throughout this paper, let k be an algebraically closed field of characteristic zero.

Definition 3.1. — *Let $\mathfrak{a}$ be an ideal contained in the maximal ideal $\mathfrak{m}_k$ of a power series ring $R = k[[x_1, \ldots, x_n]]$. Let $\ : Y \to X = \operatorname{Spec} R$ be a log resolution of $\mathfrak{a}$, and write $\mathfrak{a} \cdot O_Y = O_Y(-\sum a_i E_i)$ and $K_{Y/X} = K_Y \otimes {}^*(K_X)^*$ equal to $O_Y(\sum_i e_i E_i)$. Define the log canonical threshold of the ideal $\mathfrak{a}$ at the origin in X to be*

$$\operatorname{lct}_0(\mathfrak{a}) = \min \left\{ \frac{e_i + 1}{a_i} \right\}.$$

The log canonical threshold of an ideal is independent of the choice of log resolution, by Theorem 9.2.18 in Lazarsfeld [Lazarsfeld].

For analytic functions $f_1, \ldots, f_r$ which vanish at the origin in $\mathbf{C}^n$, the log canonical threshold of the ideal $(f_1, \ldots, f_r)$ is equal to the supremum of the numbers s such that $(\max\{|f_1|, \ldots, |f_r|\})^{-s}$ is L^2 near the origin, by the same argument as for a single function in Section 2.

For example, the ideal $(x_1, \ldots, x_n)$ in $\mathbf{C}^n$ has log canonical threshold at the origin equal to n. (It suffices to blow $\mathbf{C}^n$ up at the origin; the exceptional divisor E_1 has $e_1 = n - 1$.) More examples are given by the following version of the Thom-Sebastiani lemma:

Lemma 3.2. — *Let $\mathfrak{a} \subset k[[x_1, \ldots, x_m]]$ and $\mathfrak{b} \subset k[[y_1, \ldots, y_n]]$ be ideals in disjoint sets of variables. Then the ideal $\mathfrak{c} = (\mathfrak{a}, \mathfrak{b}) \subset k[[x_1, \ldots, x_m, y_1, \ldots, y_n]]$ has*

$$\operatorname{lct}_0(\mathfrak{c}) = \operatorname{lct}(\mathfrak{a}) + \operatorname{lct}(\mathfrak{b}).$$

This follows by the same argument as for a single function [Ksing, Proposition 8.21]. For example, Lemma 3.2 gives the calculation

$$\operatorname{lct}_0(x_1^{a_1}, \ldots, x_n^{a_n}) = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

For any ideal $\mathfrak{a}$ in a power series ring $k[[x_1, \ldots, x_n]]$, it is immediate from the definition that $\operatorname{lct}(\mathfrak{a}^m) = \operatorname{lct}(\mathfrak{a})/m$. Therefore it makes sense to define $\operatorname{lct}(\mathfrak{a}^q)$ for any positive real number q, as $\operatorname{lct}(\mathfrak{a}^q) = \operatorname{lct}(\mathfrak{a})/q$. This agrees with the definition of the log canonical threshold if we think of the pullback of the object $\mathfrak{a}^q$ to Y as the $\mathbf{R}$-divisor $-\sum q a_i E_i$.

Using pullbacks in this way, we can more generally define the log canonical threshold of $\mathfrak{a}^q \cdot \mathfrak{b}^r$ for any ideals $\mathfrak{a}$ and $\mathfrak{b}$ and any positive real numbers q and r.

Definition 3.3. — *For a natural number n, let $\mathcal{T}_n^{\mathrm{sm}}$ be the set of all log canonical thresholds of ideals $\mathfrak{a}$ contained in the maximal ideal of $k[[x_1, \dots, x_n]]$.*

The set $\mathcal{T}_n^{\mathrm{sm}}$ is independent of the choice of algebraically closed field k of characteristic zero [DM, Proposition 3.3]. It contains the set $\mathcal{HT}_n$ of log canonical thresholds of functions on a smooth n-fold, and so the ACC property for $\mathcal{T}_n^{\mathrm{sm}}$ will imply it for $\mathcal{HT}_n$. The definition of log canonical thresholds shows that $\mathcal{T}_n^{\mathrm{sm}}$ is a subset of the rationals contained in the interval $[0, n]$.

4. APPROXIMATION OF THE LOG CANONICAL THRESHOLD

In this section we state a crucial lemma, Lemma 4.1, due to Kollár (in the main case of principal ideals) [KACC, Proposition]. The lemma says that under certain conditions, the log canonical threshold of an ideal is not changed if we change the ideal by adding terms of high degree.

Lemma 4.1. — *Let $\mathfrak{a}$ be an ideal in $R = k[[x_1, \dots, x_n]]$ which is contained in the maximal ideal $\mathfrak{m}_k$. Suppose that the log canonical threshold of $\mathfrak{a}$ is computed by some divisor E on a log resolution of $X = \operatorname{Spec} R$ whose image in X is the origin. If $\mathfrak{b}$ is an ideal such that $\mathfrak{a} + \mathfrak{q} = \mathfrak{b} + \mathfrak{q}$, where $\mathfrak{q} = \{h \in k[[x_1, \dots, x_n]] : \operatorname{ord}_E(h) > \operatorname{ord}_E(\mathfrak{a})\}$, then $\operatorname{lct}(\mathfrak{b}) = \operatorname{lct}(\mathfrak{a})$.*

The proof as simplified by de Fernex, Ein, and Mustaţă [DEM, Corollary 3.5] uses only the Connectedness theorem of Shokurov and Kollár [Ksing, Theorem 7.4]. Here is the case we need.

Theorem 4.2 (Connectedness theorem, smooth case). — *Let g be a complex analytic function on a neighborhood X of the origin in $\mathbf{C}^n$. Let $\ : Y \to X$ be a log resolution of g. Write $g \cdot O_Y = O_Y(-\sum a_i E_i)$ and $K_{Y/X} = O_Y(\sum_i e_i E_i)$. Let c be a positive rational number. Then the union of the divisors E_i such that $(e_i + 1)/ca_i \leq 1$ is connected in a neighborhood of $\ ^{-1}(0)$.*

The Connectedness theorem itself is an ingenious but quick consequence of the Kawamata-Viehweg vanishing theorem, hence ultimately of the Kodaira vanishing theorem. It is striking that the only deep ingredients of the proof of ACC for smooth varieties are resolution of singularities and the Connectedness theorem, both of which were available by 1992.

5. GENERIC LIMITS OF IDEALS

We define in this section a “weak limit” of an arbitrary sequence of ideals in a power series ring, called the generic limit. The generic limit was first defined by de Fernex and Mustaţă using ultraproducts [DM]. The construction was then simplified by Kollár to use only elementary algebraic geometry [KACC]; we follow the exposition by de Fernex, Ein, and Mustaţă [DEM]. The generic limit shows that the limit of a sequence of log canonical thresholds, arising from an arbitrary sequence of ideals, must be equal to the log canonical threshold of some other ideal.

Let $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series over an algebraically closed field k, and let $\mathfrak{m}_k$ be its maximal ideal $(x_1, \ldots, x_n)$. For a field extension $k \subset K$, define R_K to mean $K[[x_1, \ldots, x_n]]$ and $\mathfrak{m}_K = \mathfrak{m} \cdot R_K$.

For each natural number d, we can identify ideals in $R/\mathfrak{m}^d$ with ideals in R that contain $\mathfrak{m}^d$. Let H_d be the Hilbert scheme of ideals in $R/\mathfrak{m}^d$, which is a closed subscheme of a finite union of Grassmannians (the Grassmannian of linear subspaces of codimension j, for each $0 \leq j \leq \dim_k(R/\mathfrak{m}^d)$). Mapping an ideal in $R/\mathfrak{m}^d$ to its image in $R/\mathfrak{m}^{d-1}$ gives a surjection $t_d : H_d(k) \to H_{d-1}(k)$, which is not a morphism of schemes. But it is a morphism on the subset of $H_d(k)$ corresponding to ideals in $R/\mathfrak{m}^d$ whose image in $R/\mathfrak{m}^{d-1}$ has a given codimension. Therefore, for each closed subvariety Z of H_d (note that we understand a subvariety to be irreducible), t_d induces a rational map $Z \dashrightarrow H_{d-1}$.

Let $\mathfrak{a}_0, \mathfrak{a}_1, \ldots$ be a sequence of ideals in R. I claim that we can choose a sequence of closed subvarieties $Z_d \subset H_d$ such that: (1) t_d induces a dominant rational map from Z_d to Z_{d-1} for each $d \geq 0$, and (2) for each d there are infinitely many i such that the ideal $\mathfrak{a}_i + \mathfrak{m}^d$ corresponds to a point in Z_d, and Z_d is minimal among subvarieties of H_d with this property.

Indeed, suppose inductively that we have constructed $Z_0 \ldots, Z_{d-1}$ with properties (1) and (2). In particular, there is an infinite set S of natural numbers i such that the ideal $\mathfrak{a}_i + \mathfrak{m}^{d-1}$ corresponds to a point in Z_{d-1}. For i in S, the ideal $\mathfrak{a}_i + \mathfrak{m}^d$ corresponds to a point in $t_d^{-1}(Z_{d-1})$, and so there is a closed subvariety Y of H_d (say, the closure of an irreducible component of this inverse image) which contains the point corresponding to $\mathfrak{a}_i + \mathfrak{m}^d$ for infinitely many i in S. Let Z_d be a subvariety of Y which is minimal among closed subvarieties that contain the point $\mathfrak{a}_i + \mathfrak{m}^d$ for infinitely many i in S; of course this exists, since a descending chain of subvarieties of H_d has length at most the dimension of H_d. In particular, these points $\mathfrak{a}_i + \mathfrak{m}^d$ are Zariski dense in Z_d by this minimality. By construction, t_d maps a dense open subset of Z_d into Z_{d-1}, and this rational map is dominant by the minimality property of Z_{d-1}. Thus we have constructed $Z_0, Z_1, Z_2, \ldots$ satisfying (1) and (2) by induction.

The dominant rational maps $\cdots \dashrightarrow Z_1 \dashrightarrow Z_0$ induce inclusions of function fields $k(Z_0) \subset k(Z_1) \subset \cdots$. Let K be the union of this sequence of fields. For each $d \geq 0$, the morphism $\mathrm{Spec}(K) \to Z_d \subset H_d$ corresponds to an ideal $\mathfrak{a}'_d$ in R_K containing $\mathfrak{m}_K^d$.

$\mathrm{lct}((\mathfrak{a}+\mathfrak{m}_K^d)\cdot\mathfrak{m}_K^q) = \mathrm{lct}((\mathfrak{a}_{i_j}+\mathfrak{m}_k^d)\cdot\mathfrak{m}_k^q)$. Therefore,

$$\begin{aligned}|\mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^q) - \mathrm{lct}(\mathfrak{a}_{i_d}\cdot\mathfrak{m}_k^q)| &\leq |\mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^q) - \mathrm{lct}((\mathfrak{a}+\mathfrak{m}_K^d)\cdot\mathfrak{m}_K^q)| \\ &\quad + |\mathrm{lct}((\mathfrak{a}_{i_d}+\mathfrak{m}_k^d)\cdot\mathfrak{m}_k^q) - \mathrm{lct}(\mathfrak{a}_{i_d}\cdot\mathfrak{m}_k^q)| \\ &\leq \frac{2n}{d} \\ &\leq \frac{2n}{j}.\end{aligned}$$

The statement on limits follows immediately. $\square$

6. ACC FOR LOG CANONICAL THRESHOLDS ON SMOOTH VARIETIES

In this section, we present the proof by de Fernex, Ein, and Mustaţă of the ACC property for log canonical thresholds on smooth varieties [DEM]. We also prove the smooth case of Kollár's Accumulation Conjecture, proved by Kollár [KACC].

THEOREM 6.1. — *For each $n \geq 0$, the set $\mathcal{T}_n^{\mathrm{sm}} \subset \mathbf{Q} \cap [0, n]$ satisfies the ascending chain condition, and its set of accumulation points is $\mathcal{T}_{n-1}^{\mathrm{sm}}$.*

A first easy step is to replace an ideal by another ideal with the same log canonical threshold, such that this log canonical threshold is computed by a divisor with a zero-dimensional center, as follows (Lemma 5.2 in [DEM]).

LEMMA 6.2. — *Let $\mathfrak{a}$ be an ideal contained in the maximal ideal $\mathfrak{m}_K$ of $K[[x_1,\dots,x_n]]$. Let $q = \max\{t \geq 0 : \mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^t) = \mathrm{lct}(\mathfrak{a})\}$.*

(i) *The nonnegative number q is rational.*

(ii) *We have $\mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^q) = \mathrm{lct}(\mathfrak{a})$, and this log canonical threshold is computed by a divisor with center on $X = \mathrm{Spec}\,K[[x_1,\dots,x_n]]$ equal to the origin.*

(iii) *The number q is zero if and only if $\mathrm{lct}(\mathfrak{a})$ is computed by a divisor with center on X equal to the origin.*

PROOF. Let $\pi : Y \to X$ be a log resolution of $\mathfrak{a}\cdot\mathfrak{m}_K$, and write $\mathfrak{a}\cdot O_Y = O_Y(-\sum a_iE_i)$, $\mathfrak{m}_K\cdot O_Y = O_Y(-\sum b_iE_i)$, and $K_{Y/X} = K_Y \otimes \pi^*(K_X)^*$ equal to $O_Y(\sum_i e_iE_i)$. Let I be the set of those i such that E_i has center equal to the origin, that is, such that $b_i > 0$.

Let $c = \mathrm{lct}(\mathfrak{a})$. The log canonical threshold decreases (that is, the singularity gets worse) as we increase a subscheme of X, and so $\mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^t) \leq c$ for all $t \geq 0$. Furthermore, $\mathrm{lct}(\mathfrak{a}\cdot\mathfrak{m}_K^t) \geq c$ if and only if

$$e_i + 1 \geq c(a_i + tb_i)$$

for all i. If i is not in I, then $b_i = 0$ and this inequality holds for all t. We conclude that

$$q = \min\left\{\frac{e_i + 1 - ca_i}{cb_i} : i \in I\right\}.$$

Thus q is rational. Moreover, there is an $i \in I$ such that this minimum is achieved; then E_i computes $\mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q)$, and E_i has center on X equal to the origin. The statement in (iii) is clear. □

Proof of Theorem 6.1. Let c_i be a strictly increasing or strictly decreasing sequence in $\mathcal{T}_n^{\mathrm{sm}}$. We will get a contradiction if the sequence is strictly increasing, while we will show that the limit of the sequence is in $\mathcal{T}_{n-1}^{\mathrm{sm}}$ if the sequence is strictly decreasing. Conversely, every number in $\mathcal{T}_{n-1}^{\mathrm{sm}}$ is the limit of a strictly decreasing sequence in $\mathcal{T}_n^{\mathrm{sm}}$. Indeed, for an ideal $\mathfrak{a} \subset k[[x_1, \dots, x_{n-1}]]$, we have $\mathrm{lct}_0((\mathfrak{a}, x_n^d)) = \mathrm{lct}_0(\mathfrak{a}) + 1/d$ by Lemma 3.2.

Let $c = \lim_i c_i$; the limit is finite, since $\mathcal{T}_n^{\mathrm{sm}}$ is contained in $[0, n]$. For each i, there is an ideal $\mathfrak{a}_i$ in $R = k[[x_1, \dots, x_n]]$ with log canonical threshold at the origin equal to c_i. Let $\mathfrak{a}$ be a generic limit of the sequence $\mathfrak{a}_i$ (Definition 5.1), with $\mathfrak{a} \subset K[[x_1, \dots, x_n]]$. By construction of the generic limit, $\mathfrak{a}$ is contained in the maximal ideal $\mathfrak{m}_K$. By Corollary 5.3, the limiting ideal $\mathfrak{a}$ has log canonical threshold equal to c. If $c = 0$, then the sequence c_i must be strictly decreasing rather than strictly increasing. We must have $n > 0$ (as $\mathcal{T}_0^{\mathrm{sm}} = \{0\}$), and so 0 is in $\mathcal{T}_{n-1}^{\mathrm{sm}}$ as we want. Thus we can assume that $c > 0$. In particular, the ideal $\mathfrak{a}$ is not zero.

Let q be the rational number attached to $\mathfrak{a}$ by Lemma 6.2. We have

$$\mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q) = \mathrm{lct}(\mathfrak{a}).$$

We also have the trivial inequality

$$\mathrm{lct}(\mathfrak{a}_i \cdot \mathfrak{m}_k^q) \leq \mathrm{lct}(\mathfrak{a}_i)$$

for all i. In particular, when c_i is a strictly increasing sequence, we have $\mathrm{lct}(\mathfrak{a}_i \cdot \mathfrak{m}_k^q) < \mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q)$ for all i.

By the choice of q, $\mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q)$ is computed by a divisor E which lies over the origin in $X = \mathrm{Spec}\, K[[x_1, \dots, x_n]]$. For any $d > \mathrm{ord}_E(\mathfrak{a} \cdot \mathfrak{m}_K^q)$, the continuity lemma (Lemma 4.1) gives that

$$\mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q) = \mathrm{lct}((\mathfrak{a} + \mathfrak{m}_K^d) \cdot \mathfrak{m}^q),$$

and E computes both log canonical thresholds.

By Lemma 5.2, there is an integer $d > \mathrm{ord}_E(\mathfrak{a} \cdot \mathfrak{m}_K^q)$ and an infinite set $S \subset \mathbf{N}$ such that, for every $i \in S$, we have $\mathrm{lct}((\mathfrak{a} + \mathfrak{m}_K^d) \cdot \mathfrak{m}_K^q) = \mathrm{lct}((\mathfrak{a}_i + \mathfrak{m}_k^d) \cdot \mathfrak{m}_k^q)$. The lemma also gives a divisor E_i over $\mathrm{Spec}\, k[[x_1, \dots, x_n]]$ which computes $\mathrm{lct}((\mathfrak{a}_i + \mathfrak{m}_k^d) \cdot \mathfrak{m}_k^q)$ and which has

$$\mathrm{ord}_{E_i}((\mathfrak{a}_i + \mathfrak{m}_k^d) \cdot \mathfrak{m}_k^q) = \mathrm{ord}_E((\mathfrak{a} + \mathfrak{m}_K^d) \cdot \mathfrak{m}_K^q) = \mathrm{ord}_E(\mathfrak{a} \cdot \mathfrak{m}_K^q).$$

By our inequality on d,

$$\mathrm{ord}_{E_i}(\mathfrak{m}_k^d) \geq d > \mathrm{ord}_E(\mathfrak{a} \cdot \mathfrak{m}_K^q) = \mathrm{ord}_{E_i}((\mathfrak{a}_i + \mathfrak{m}_k^d) \cdot \mathfrak{m}_k^q).$$

By the continuity lemma (Lemma 4.1), for each i in the infinite set S, we have

$$\mathrm{lct}(\mathfrak{a}_i \cdot \mathfrak{m}_k^q) = \mathrm{lct}((\mathfrak{a}_i + \mathfrak{m}_k^d) \cdot \mathfrak{m}_k^q) = \mathrm{lct}((\mathfrak{a} + \mathfrak{m}_K^d) \cdot \mathfrak{m}_K^q) = \mathrm{lct}(\mathfrak{a} \cdot \mathfrak{m}_K^q).$$

If c_i is a strictly increasing sequence, this equality contradicts the strict inequality found earlier. Thus we have proved the ACC property for log canonical thresholds on smooth varieties.

Finally, suppose that $c_i = \mathrm{lct}(\mathfrak{a}_i)$ is a strictly decreasing sequence. We have shown that the sequence $\mathrm{lct}(\mathfrak{a}_i \cdot \mathfrak{m}_k^q)$ has infinitely many terms that are equal, and so q is not zero. Equivalently, by Lemma 6.2, $\mathrm{lct}(\mathfrak{a})$ is not computed by any divisor whose center in X is the origin.Therefore, if F is a divisor over $\mathrm{Spec}\, K[[x_1, \ldots, x_n]]$ that computes the log canonical threshold of $\mathfrak{a}$, then the center of F in $\mathrm{Spec}\, K[[x_1, \ldots, x_n]]$ has positive dimension. After localizing at the generic point of this center, we deduce that $\mathrm{lct}(\mathfrak{a})$ is in $\mathcal{T}_{n-1}^{\mathrm{sm}}$. (We use here that the completed local ring of a regular scheme containing a field at a codimension r integral subscheme W is isomorphic to the power series ring $L[[x_1, \ldots, x_r]]$, where L is the function field of W.) Since the numbers $\mathrm{lct}(\mathfrak{a}_i)$ converge to $\mathrm{lct}(\mathfrak{a})$, we have shown that all accumulation points of $\mathcal{T}_n^{\mathrm{sm}}$ lie in $\mathcal{T}_{n-1}^{\mathrm{sm}}$. □

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