Séminaire BOURBAKI Avril 2011
63ème année, 2010-2011, n$^{\text{o}}$ 1037

A PROOF OF THE ANDRÉ-OORT CONJECTURE VIA MATHEMATICAL LOGIC [after Pila, Wilkie and Zannier]

by **Thomas SCANLON**

INTRODUCTION

Extending work of Bombieri and Pila on counting lattice points on convex curves [3], Pila and Wilkie proved a strong counting theorem on the number of rational points in a more general class of sets *definable* in an o-minimal structure on the real numbers [37]. Following a strategy proposed by Zannier, the Pila-Wilkie upper bound has been leveraged against Galois-theoretic lower bounds in works by Daw, Habegger, Masser, Peterzil, Pila, Starchenko, Yafaev and Zannier [6, 18, 25, 31, 36, 38] to prove theorems in diophantine geometry to the effect that for certain algebraic varieties the algebraic relations which may hold on its "special points" are exactly those coming from "special varieties". Of these results, Pila's unconditional proof of the André-Oort conjecture for the j-line is arguably the most spectacular and will be the principal object of this resumé. Readers interested in a survey with more details about some of the other results along these lines, specifically the Pila-Zannier reproof of the Manin-Mumford conjecture and the Masser-Zannier theorem about simultaneous torsion in families of elliptic curves, may wish to consult my notes for the Current Events Bulletin lecture [43].

Acknowledgements. I wish to thank M. Aschenbrenner, J. Pila and U. Zannier for their advice and especially for suggesting improvements to this text.

1. STATEMENT OF THE ANDRÉ-OORT CONJECTURE

The collection of theorems and conjectures broadly known under the rubric of the André-Oort conjecture arose from a conjecture proposed by André about curves in Shimura varieties [1] and a related conjecture of Oort that a subvariety of a moduli space of principally polarized abelian varieties which contains a Zariski dense set of moduli points of abelian varieties with complex multiplication must be a variety of Hodge type [29]. The assertion now generally regarded as *the* André-Oort conjecture takes as its starting point the theory of Shimura varieties as presented in terms of Deligne's Shimura data and predicts that the Zariski closure of a set of special points in a Shimura variety must be a finite union of varieties of Hodge type. The full André-Oort conjecture paints a beautiful picture of the way in which the diophantine geometry of

Shimura varieties reflects the presentation of these varieties as quotients of homogeneous spaces by the action of an arithmetic group. However, since too much theoretical overhead is required to correctly state this conjecture and even more is required to do the subject justice, not to mention the fact that Noot's excellent survey [28] is already available, we shall restrict to the case considered in [36].

Let us recall some of the classical complex analytic theory of elliptic curves as one would find in such sources as [42] or [45].

Let $\mathfrak{h} := \{z \in \mathbb{C} \;:\; \mathrm{Im}(z) > 0\}$ be the upper half plane consisting of those complex numbers with positive imaginary part. For $\tau \in \mathfrak{h}$, let $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the one-dimensional complex torus (which is necessarily an elliptic curve, that is, a connected, one-dimensional algebraic group) obtained as the quotient of the additive group of the complex numbers by the lattice generated by 1 and τ. The group $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$ via fractional linear transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z := \frac{az+b}{cz+d}$. A simple computation shows that $E_\tau \cong E_\sigma$ just in case there is some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $\gamma \cdot \tau = \sigma$. Hence, as a set, we may identify the set of isomorphism classes of complex elliptic curves with $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$. The analytic j-function (or "modular function") $j : \mathfrak{h} \to \mathbb{C}$ is a surjective holomorphic function which is exactly $\mathrm{PSL}_2(\mathbb{Z})$-invariant in the sense that $j(\tau) = j(\sigma)$ if and only if $\sigma = \gamma \cdot \tau$ for some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$. Thus, using j we may identify $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$ with $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$ and we say that a number $\xi \in \mathbb{C}$ is the j-invariant of an elliptic curve E if there is some $\tau \in \mathfrak{h}$ for which $\xi = j(\tau)$ and $E \cong E_\tau$.

From the general theory of covering spaces, one sees that

$$\mathrm{Hom}(E_\tau, E_\sigma) = \{\lambda \in \mathbb{C} \;:\; \lambda(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + \mathbb{Z}\sigma\}.$$

Specializing to the case of $\tau = \tau'$ one sees that the endomorphism ring of E_τ is strictly larger than $\mathbb{Z}$ just in case τ is a quadratic imaginary number. In this case we say that E_τ has complex multiplication or that it is a CM-elliptic curve. From the above considerations, we see that a number $\xi \in \mathbb{A}^1(\mathbb{C})$ is the j-invariant of a CM-elliptic curve just in case ξ is the value of j on a quadratic imaginary number. In this way, we may regard the moduli points of CM-elliptic curves as the special values of the modular function. We say that a point $(\xi_1, \ldots, \xi_n) \in \mathbb{A}^n(\mathbb{C})$ is a *special point* just in case each ξ is a CM-moduli point.

For a positive integer $N \in \mathbb{Z}_+$ the N^{th} Hecke correspondence is the following set

$$T_N(\mathbb{C}) := \{(j(\tau), j(N\tau)) \;:\; \tau \in \mathfrak{h}\}.$$

The Hecke correspondence T_N is actually an algebraic subvariety of $\mathbb{A}^2$ defined by the vanishing of the so-called N^{th} modular polynomial $\Phi_N(x, y)$. From the definition of T_N it is obvious that $T_N(\mathbb{C})$ contains a Zariski dense set of special points as if τ is a quadratic imaginary number, then so is $N\tau$. Pila's theorem asserts that in a precise sense these are the only interesting varieties which contain a dense set of special points.

Theorem 1.1. — *Let $n \in \mathbb{Z}_+$ be a positive integer and $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ an irreducible subvariety of affine n-space over the complex numbers. If X contains a Zariski dense set of special points, then X is a special variety. That is, it is a component of a variety* *defined by equations of the form* $\Phi_M(x_i, x_j) = 0$ *and* $x_\ell = \xi$ *where* Φ_M *is a modular polynomial and* ξ *is a special point.*

Remark 1.2. — Theorem 1.1 is not the strongest theorem proven in [36]. Using the methods outlined in this survey, Pila proved some cases of Pink's generalization of the André-Oort conjecture to mixed Shimura varieties in which the ambient variety is taken to be a product of a finite sequence of curves where each factor is a modular curve, an elliptic curve or the multiplicative group.

Remark 1.3. — Approximations to and conditional generalizations of Theorem 1.1 were proven some time ago. Restricting to the case that X is a curve, Edixhoven already proved Theorem 1.1 under the hypothesis of the generalized Riemann hypothesis in [13, 14] while André shortly thereafter gave an unconditional proof [2]. Edixhoven and Yafaev proved the André-Oort conjecture allowing the ambient variety to be an arbitrary Shimura variety but taking the subvariety X to be a curve under an hypothesis about constancy of the Hodge structure in [15]. Yafaev then proved the André-Oort conjecture under the Generalized Riemann Hypothesis for CM-fields where the subvariety is again a curve. In more recent work, building on results of Ullmo and Yafaev [49], Klingler and Yafaev [22] have proven the full André-Oort conjecture under either the technical hypothesis of [15] or under GRH. Working locally, Moonen proved a p-adic analogue of the André-Oort conjecture for moduli spaces of abelian varieties [27].

All of the known proofs share a common fundamental structure. Geometric reasoning leads to upper bounds on the number of special points lying on a given non-special variety outside of its positive dimensional special subvarieties. Arguments of an analytic number theoretic nature combined with some Galois-theoretic considerations produce lower bounds which outstrip the upper bounds if there are too many special points. In the papers preceding [36], the upper bounds generally come from intersection theory whereas in [36], the upper bounds come from the Pila-Wilkie counting theorem for o-minimal theories.

2. FIRST STEPS TOWARDS THE PROOF

The reader is likely familiar with the ploy of introducing the theory of elliptic curves via complex analysis only to shift the perspective to algebraic number theory and algebraic geometry as soon as diophantine issues arise. However, in the case of Theorem 1.1, the proof proceeds through the complex analytic presentation.

Let us begin with $X \subseteq \mathbb{A}^n$ an affine algebraic variety and let us fix some polynomials $F_1(x_1, \ldots, x_n), \ldots, F_\ell(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ for which

$$X(\mathbb{C}) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n(\mathbb{C}) \ : \ F_j(a_1, \ldots, a_n) = 0 \text{ for } i \leq \ell\}.$$

We wish to describe the set of special points on X. That is, we wish to describe the set of n-tuples of quadratic imaginary numbers $(\tau_1, \ldots, \tau_n)$ for which $F_1(j(\tau_1), \ldots, j(\tau_n)) = \cdots = F_\ell(j(\tau_1), \ldots, j(\tau_n)) = 0$.

Consider the following real analytic set

$$\begin{aligned}\mathfrak{X} \; := \; & \{(x_1, \cdots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^n \times (\mathbb{R}_+)^n \; : \\ & F_t(j(x_1 + i\sqrt{y_1}), \ldots, j(x_n + i\sqrt{y_n})) = 0 \text{ for } t \leq \ell\}.\end{aligned}$$

The set of special points of $X(\mathbb{C})$ is the image of the set of rational points on $\mathfrak{X}$ under the map

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (j(x_1 + i\sqrt{y_1}), \ldots, j(x_n + i\sqrt{y_n})).$$

Thus, we have succeeded in reducing the admittedly difficult André-Oort conjecture to the intractable problem of describing the set of rational points on a real analytic variety. Since each subset of $\mathbb{Z}^{2n}$ may be realized as the zero set of a real analytic function, knowing merely that $\mathfrak{X}$ is real analytic yields no useful information. On the other hand, even knowing that $\mathfrak{X}$ is defined by particularly simple equations does not seem to help as, for instance, the problem of describing the rational points on an algebraic variety is notoriously difficult.

The strength of this reduction comes from $\mathfrak{X}$ avoiding these extremes of a general real analytic variety on one hand and of an algebraic variety on the other. The geometry of $\mathfrak{X}$ is simple in that, at least when it is restricted to an appropriate fundamental domain, it is definable in an o-minimal expansion of the real field. For such sets, the counting theorem of Pila and Wilkie gives subpolynomial (in a bound on the height) bounds for the number rational points lying in the set provided that one excludes those points lying on semi-algebraic curves.

3. INTERLUDE ON O-MINIMALITY

O-minimality is a logical condition isolated by van den Dries [8] from which the theory of semi-algebraic geometry may be developed axiomatically, and ultimately, generalized. In order to express the definition of o-minimality we require some terminology from mathematical logic and I would argue that to appreciate the strength of o-minimality one should approach the subject with a sensibility informed by logic. However, I shall keep the logical apparatus to a minimum. The reader desiring a fuller introduction to the theory of o-minimality would do well to read Wilkie's survey [53] or van den Dries' book [10].

Definition 3.1. — *An o-minimal structure is a structure $(R, <, \ldots)$ in the sense of first-order logic for which $<$ is a linear order on R and the ellipses indicate that there may be other distinguished functions and relations for which every definable (with parameters) subset of R is a finite union of points and intervals.*

For the purposes of this survey it (almost) suffices to consider only the case where the underlying ordered set is the set of real numbers with the usual ordering and the extra structure consists of a set of distinguished functions $f : \mathbb{R}^n \to \mathbb{R}$. I say "almost" as even if one is exclusively concerned with real geometry, it is often useful to apply arguments of a non-standard analytic character for which the more general notion of an o-minimal structure is crucial. Indeed, such arguments are front and center in the proof of the counting theorem in [37].

Definition 3.2. — *Suppose that for each natural number n we are given a set $\mathcal{F}_n$ of functions $f : \mathbb{R}^n \to \mathbb{R}$. By $\mathbb{R}_{\mathcal{F}}$ we mean the structure whose underlying set is $\mathbb{R}$, whose order is the usual order on the real numbers, and which has the distinguished function $f : \mathbb{R}^n \to \mathbb{R}$ for each $f \in \mathcal{F}_n$.*

Given such a structure $\mathbb{R}_{\mathcal{F}}$ by a *basic* or *atomic* definable set in $\mathbb{R}^n$ we mean a set of the form

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)\}$$

or

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ f(x_1, \ldots, x_n) < g(x_1, \ldots, x_n)\}$$

where f and g are functions of n variables built from the coordinate functions, constant functions and the distinguished functions in $\mathcal{F}$ via appropriate compositions. The class of all definable sets is the smallest collection of subsets of $\mathbb{R}^n$ (for various n) closed under taking finite Boolean combinations and images under the coordinate projections $\mathbb{R}^{n+1} \to \mathbb{R}^n$ given by $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$.

Perhaps, some examples are in order.

Example 3.3. — Taking $\mathcal{F}$ to consist of all polynomials in any number of variables over $\mathbb{R}$, it follows from work of Tarski on the decidability of Euclidean geometry [47] that every definable set is semi-algebraic, that is, a finite Boolean combination of sets defined by conditions of the form $f(x_1, \ldots, x_n) > 0$ where $f \in \mathbb{R}[x_1, \ldots, x_n]$. Since a polynomial in one variable changes sign only finitely many times, it follows that $\mathbb{R}_{\mathcal{F}}$ is o-minimal.

Example 3.4. — Note that $\{x \in \mathbb{R} \ : \ \sin(x) = 0\} = \mathbb{Z}\pi$ is an infinite, discrete set and as such cannot be expressed as a finite union of points and intervals. Hence, $\mathbb{R}_{\{\sin\}}$ is not o-minimal.

Example 3.5. — We say that $f : \mathbb{R}^n \to \mathbb{R}$ is a *restricted analytic function* if there is a neighborhood $U \supseteq [-1,1]^n$ of the n-cube $[-1,1]^n$ and a real analytic function $\widetilde{f} : U \to \mathbb{R}$ for which $f(x) = \widetilde{f}(x)$ for $x \in [-1,1]^n$ and $f(x) = 0$ for $x \in \mathbb{R}^n \smallsetminus [-1,1]^n$. If we let $\mathcal{F}$ consist of all polynomials over $\mathbb{R}$ and all restricted analytic functions, then van den Dries observed [9] that the o-minimality of $\mathbb{R}_{\mathcal{F}}$ (usually denoted as $\mathbb{R}_{an}$) follows as a consequence of results of Gabrielov [16] on semi-analytic geometry. Thereafter, Denef and van den Dries [7] presented a more direct proof of the o-minimality of $\mathbb{R}_{an}$. The key technical observation required for their proof is that the Weierstrass Preparation and

Division Theorems permit one to replace conditions on the sign of an analytic function of a single variable over a closed interval with the same conditions on an associated polynomial.

Example 3.6. — Extending work of Khovanski on so-called fewnomials [21], Wilkie [51] showed that if $\mathcal{F}$ contains all the polynomials over $\mathbb{R}$ together with the real exponential function, then $\mathbb{R}_{\exp} := \mathbb{R}_{\mathcal{F}}$ is o-minimal. Wilkie extended this result to obtain the stronger theorem that the expansion of the real field by all functions which satisfy iterated Pfaffian differential equations is o-minimal [52].

Example 3.7. — Amalgamating the last two examples so that $\mathcal{F}$ consists of all restricted analytic functions, all polynomials, and the real exponential function we obtain $\mathbb{R}_{an,\exp}$ which van den Dries and Miller proved to be o-minimal [12]. In subsequent work, van den Dries, Macintyre, and Marker analyzed the definable sets in $\mathbb{R}_{an,\exp}$ through the study of generalized power series models [11]. Thereafter, Speissegger showed that if $\mathbb{R}_{\mathcal{F}}$ is an o-minimal structure and $f : \mathbb{R} \to \mathbb{R}$ is a function which satisfies a Pfaffian differential equation over $\mathbb{R}_{\mathcal{F}}$, that is, there is some $G(x, y) \in \mathcal{F}_2$ for which f satisfies the differential equation $Y' = G(x, Y)$, then the structure obtained by adjoining f to $\mathcal{F}_1$ is still o-minimal [46].

The great virtue of the notion of o-minimality is that from the hypothesis about the simplicity of the definable subsets of the line one may deduce strong regularity results about the definable sets in higher dimensions. The fundamental theorem of o-minimality is the cell decomposition theorem which was first proven by van den Dries under the hypothesis that the underlying ordered set is $(\mathbb{R}, <)$ [8] and in full generality by Knight, Pillay and Steinhorn [23, 39, 40].

Definition 3.8. — *Given an o-minimal structure $(R, <, \ldots)$ we define the class of cells in R^n and their dimension by recursion on n. When $n = 1$, singleton sets $\{a\}$ and intervals (a, b) where we allow the possibility that $a = -\infty$ and that $b = \infty$ are cells. Their dimensions are 0 and 1, respectively. If $X \subseteq R^n$ is a cell and $f : X \to R$ is a continuous (with respect to the order topology), definable function (in the sense that its graph is a definable set), then the graph of f is a cell in R^{n+1} with the same dimension as that of X. If $g : X \to R$ is another continuous, definable function on X for which $f(x) < g(x)$ for every $x \in X$, then the parametrized interval*

$$(f, g)_X := \{(x, y) \in R^n \times R \ : \ x \in X \ \& \ f(x) < y < g(x)\}$$

is a cell of dimension one more than that of X. Likewise, infinite intervals $(-\infty, f)_X$ and $(f, \infty)_X$ (with the obvious definitions) are cells also of dimension one more than that of X.

Theorem 3.9. — *If $(R, <, \ldots)$ is an o-minimal structure and $X \subseteq R^n$ is definable, then there is a partition of R^n into finitely many cells so that X may be expressed as a union of some of these cells.*

PROOF (sketch) — One argues by induction on n noting that the case of $n = 1$ is exactly the definition of o-minimality and proving along the way two auxiliary results. First, if $f : R \to R$ is any definable function then R may be decomposed into finitely many points and open intervals so that on each such open interval f is strictly monotone or constant. Secondly, every definable function $f : R^n \to R$ is piecewise continuous in the sense that the domain admits a decomposition into finitely many cells for which the restriction of f is continuous. For the inductive argument one shows first that for every definable set X in R^{n+1} there is a cell decomposition of R^{n+1} compatible with X and then establishes the piecewise continuity of definable functions $f : R^{n+1} \to R$.

The key to proving piecewise monotonicity of functions of a single variable is the observation that the sets where f is locally increasing (or decreasing or constant) are definable. Using o-minimality, one shows that if the lemma failed, then there would be an open interval I on which f is never locally constant, locally increasing, or locally decreasing and then derives a contradiction to o-minimality by considering sets of the form $\{x \in I \ : \ f(x) < f(c)\}$ for some fixed $c \in I$.

Piecewise continuity of definable functions $f : R^{n+1} \to R$ is shown by observing that the set of points at which f is continuous is definable and then invoking cell decomposition to conclude that if the result were false there would be an open cell on which f is everywhere discontinuous. One then reaches a contradiction by considering the family of functions $(g_a : R^n \to R)_{a \in R}$ given by $g_a(x_1, \ldots, x_n) := f(x_1, \ldots, x_n, a)$ which we know to be piecewise continuous and the functions $(h^b : R \to R)_{b \in R^n}$ given by $h(x) := f(b, x)$ which we know to be piecewise monotone.

Finally, for $X \subseteq R^{n+1}$ a definable set, we define a sequence of (possibly partial) functions $f_m : R^n \to R$ by sending a to the m^{th} point in the boundary of $X_a := \{y \in R : (a, y) \in X\}$. Via a nontrivial argument one shows that the cardinality of the boundary of X_a is bounded. By induction, we may decompose R^n into cells on which all of the functions f_i are continuous and the truth value of conditions of the form $f_i(x) \in X_x$ or $y \in X_x$ for some $y \in (f_j(x), f_{j+1}(x))$ is constant. The cell decomposition statement for X follows. □

It is hard to overstate the importance of Theorem 3.9. For example, it follows from the cell decomposition theorem that if $\mathbb{R}_{\mathcal{F}}$ is o-minimal and $X \subseteq \mathbb{R}^{n+m}$ is a definable set, then there are only finitely many homeomorphism types amongst the sets $X_a := \{b \in R^m : (a, b) \in X\}$ as a varies through R^n. In almost every proof of a nontrivial result about sets definable in an o-minimal structure Theorem 3.9 will be invoked repeatedly. This is certainly the case for the Pila-Wilkie counting theorem.

4. COUNTING RATIONAL POINTS IN DEFINABLE SETS

Let us introduce the requisite definitions so that we may state the Pila-Wilkie counting theorem.

Definition 4.1. — *For $r \in \mathbb{Q}$ a rational number we define the height of r by $H(r) = 0$ if $r = 0$ and $H(r) = \max\{|a|, |b|\}$ if $r = \frac{a}{b}$ where a and b are coprime integers. For $r = (r_1, \ldots, r_n) \in \mathbb{Q}^n$ we define $H(r) := \max\{H(r_i) \ : \ i \leq n\}$.*

Remark 4.2. — The height function of Definition 4.1 may be smaller than the restriction of the usual height function on $\mathbb{P}^n(\mathbb{Q})$ to $\mathbb{A}^n(\mathbb{Q})$. However, it is better suited to the counting problems we shall consider.

Definition 4.3. — *For $X \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}_+$ we set*

$$X(\mathbb{Q}, t) := \{x \in X \cap \mathbb{Q}^n \ : \ H(x) \leq t\}$$

and $N(X, t) := \#X(\mathbb{Q}, t)$.

Definition 4.4. — *We say that $Y \subseteq \mathbb{R}^n$ is a semi-algebraic curve if there is a non-constant, continuous, semi-algebraic function $\gamma : (0, 1) \to \mathbb{R}^n$ whose image is Y. For $X \subseteq \mathbb{R}^n$ any set, the algebraic part of X, X^{alg}, is the union of all semi-algebraic curves contained in X. The transcendental part of X is $X^{\mathrm{tr}} := X \smallsetminus X^{\mathrm{alg}}$.*

Theorem 4.5. — *Let $X \subseteq \mathbb{R}^n$ be a definable set in some o-minimal structure $\mathbb{R}_{\mathcal{F}}$ on the real numbers. For every $\epsilon > 0$ there is a constant $C = C(\epsilon, X) > 0$ so that for all $t \in \mathbb{R}_+$ we have $N(X^{\mathrm{tr}}, t) \leq Ct^{\epsilon}$.*

Remark 4.6. — Fix a natural number D. For $X \subseteq \mathbb{R}^n$ and $t > 0$ we define

$$N(X, D, t) := \#\{(x_1, \ldots, x_n) \in X \ : \ [\mathbb{Q}(x_i) : \mathbb{Q}] \leq D \ \& \ H(x_i) \leq t \text{ for } i \leq n\}.$$

Using standard coding tricks Pila strengthens Theorem 4.5 to show that for fixed $D \in \mathbb{Z}_+$, $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure, and $\epsilon > 0$ there is a constant $C = C(\epsilon, X, D)$ so that $N(X^{\mathrm{tr}}, D, t) \leq Ct^{\epsilon}$ [35]. It is actually this form of Theorem 4.5 with $D = 2$ which we shall employ in the proof of Theorem 1.1.

Remark 4.7. — One might hope to improve the bounds in the counting theorem, say, by replacing t^{ϵ} with something like $(\log(t))^N$. Such improvements are known to fail for general o-minimal structures on the real numbers, including in $\mathbb{R}_{an}$. However, Wilkie conjectures that if $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{\exp}$, then there are constants $C, K > 0$ so that for all $t > 1$ we have $N(X^{\mathrm{tr}}, t) \leq C(\log(t))^K$. Some progress towards this conjecture has been achieved by Butler, Jones, Miller, Pila and Thomas [5, 20, 19, 34].

There are two separate parts to the proof of Theorem 4.5. First, one proves a counterpart to the cell decomposition theorem that every bounded definable set in an o-minimal structure on the real numbers may be definably parametrized by the unit ball using functions with small derivatives. Secondly, one uses these parametrizations to employ effective diophantine approximation arguments to bound the number of rational points on X but outside of the algebraic part.

Definition 4.8. — *Let $X \subseteq \mathbb{R}^n$ be a k-dimensional definable set in some o-minimal structure $\mathbb{R}_{\mathcal{F}}$ on the real numbers and let $r \in \mathbb{Z}_+$ be a positive integer. We say that $\phi = (\phi_1, \ldots, \phi_n) : (0,1)^k \to R^n$ is a partial r-parametrization of X if*

- *ϕ is definable,*
- *the range of ϕ is contained in X, and*
- *for each $i \leq n$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $|\alpha| = \sum \alpha_i \leq r$ we have $|\frac{\partial^{|\alpha|}\phi_i}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x)| \leq 1$ for all $x \in (0,1)^n$.*

*By an r-*parametrization *of X we mean a finite set S of partial r-parametrizations of X for which X is covered by the ranges of the functions in S.*

Theorem 4.9. — *If $X \subseteq [-1,1]^n$ is definable in some o-minimal structure $\mathbb{R}_{\mathcal{F}}$ on the real numbers for which $\mathcal{F}$ contains all of the polynomials and $r \in \mathbb{Z}_+$ is a positive integer, then X admits an r-parametrization.*

Remark 4.10. — While we have stated Theorem 4.9 simply for definable sets in o-minimal structures on the real numbers, its proof passes through an analysis of definable sets in arbitrary o-minimal structures.

Theorem 4.9 generalizes a theorem of Yomdin on the existence of r-parametrizations for real semi-algebraic sets [54, 55] and its proof follows Gromov's version of the proof in the semi-algebraic setting [17].

The logical structure of the proof of Theorem 4.9 is similar to that of the cell decomposition theorem (Theorem 3.9). For both of these theorems, the one-dimensional case itself is an immediate consequence of the definition of o-minimality, but to carry out the induction one performs a concurrent induction showing that definable functions have strong regularity properties. For the parametrization theorem, the rôle of the piecewise continuity theorem is played by a *reparameterization* theorem in all dimensions and in dimension one the monotonicity theorem is replaced by a very strong reparameterization in which the change of variables or the function obtained after the change of variables may be taken to be a polynomial. It is here with the reparameterization theorem that nonstandard models are crucial. By an r-reparameterization of a function f we mean a parameterization of its domain so that for each function ϕ in the parametrization the partial derivatives of $f \circ \phi$ up to order r all have absolute value bounded by some natural number. When the underlying structure is simply the ordered set of real numbers, to say that these functions are bounded by a natural number is the same as saying that they are simply bounded. For an arbitrary o-minimal structure these notions do not coincide.

Once the parameterization theorem has been established, the argument for Theorem 4.5 follows the lines of other constructive arguments bounding numbers of rational solutions and is similar in spirit to Bombieri's proof of the Mordell conjecture [4]. The key result is the following proposition the kernel of whose proof is ultimately embedded in the paper [3] and completed in [33].

Proposition 4.11. — *For $m, n, d \in \mathbb{N}$ with $m < n$ there are numbers $r \in \mathbb{Z}_+$ and $\epsilon = \epsilon(m, n, d)$ and $C = C(m, n, d)$ in $\mathbb{R}_+$ so that for any $\mathcal{C}^r$ function $\phi : (0,1)^m \to \mathbb{R}^n$ with range X and $t \geq 1$ the set $X(\mathbb{Q}, t)$ is contained in at most Ct^ϵ hypersurfaces of degree d and $\epsilon(m, n, d) \to \infty$ as $d \to \infty$.*

Proof (sketch) — We sketch the initial steps in the proof of Proposition 4.11 without expressing any of the required bounds. Take $Q_0, \ldots, Q_\ell \in (0,1)^m$ so that $\phi(Q_0), \ldots, \phi(Q_\ell)$ are distinct elements of $X(\mathbb{Q}, t)$. The condition that the points $\phi(Q_0), \ldots, \phi(Q_\ell)$ lie on some hypersurface of degree at most d may be expressed by saying that the rank of the matrix $M = (\phi(Q_i)^\beta)$ be sufficiently small. Here, we have indexed the matrix by $0 \leq i \leq \ell$ and $\beta \in \mathbb{N}^n$ with $|\beta| = \sum_{i=1}^n \beta_i \leq d$ and we have written $\phi(Q_i)^\beta$ for $\prod_{j=1}^n \phi_j(Q_i)^{\beta_j}$. Using some simple considerations of convex geometry and the bounds on the derivatives of ϕ up to order r, via Taylor expansions one bounds the size of the determinants of the appropriate minors of M. At this point we apply the observation that the set of rational numbers with bounded denominators is discrete to conclude that the determinants of these minors are actually zero. □

With Proposition 4.11 in place, Theorem 4.5 follows by induction: those exceptional hypersurfaces which have full dimensional intersection with X are part of X^{alg} and those which intersect X in a lower dimensional set contribute little to $N(X^{\mathrm{tr}}, t)$ by induction.

5. COMPLETING THE SKETCH OF THE PROOF OF THE MAIN THEOREM

Let us return to the proof of Theorem 1.1 we began to sketch in Section 2.

Our first observation is that the j-function, appropriately restricted and properly interpreted, is definable in the o-minimal structure $\mathbb{R}_{an,\exp}$. That is, identifying $\mathbb{C}$ with $\mathbb{R}^2$ we may regard j as a real analytic function from an open region in $\mathbb{R}^2$ to $\mathbb{R}^2$. The full j-function is not definable in any o-minimal structure on the real numbers [30]. One can see this, for instance, by observing that the preimage of any point is a countably infinite set which cannot be expressed as a finite union of cells. However, if D is a fundamental domain for j, for example, $D = \{\tau \in \mathfrak{h} \ : \ |\tau| \geq 1 \ \& \ \frac{-1}{2} \leq \mathrm{Re}(\tau) < \frac{1}{2}\}$, then the restriction of j to D is definable in $\mathbb{R}_{an,\exp}$. Indeed, $j(\tau) = J(e^{2\pi i \tau})$ where J is a meromorphic function on the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ having a simple pole at the origin. Again interpreting $\mathbb{C}$ as $\mathbb{R}^2$ we see that for any $r < 1$ the restriction of J to the closed disk of radius r is definable in $\mathbb{R}_{an}$. Since the image of D under the map $\tau \mapsto e^{2\pi i \tau}$ is contained in the closed disk of radius $e^{-\pi\sqrt{3}}$, we conclude that the restriction of j to D is definable in $\mathbb{R}_{an,\exp}$.

Consider now a purported counterexample to Theorem 1.1, that is, an irreducible algebraic variety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ which is not special but still contains a Zariski dense set of

n-tuples of j-invariants of CM-elliptic curves. Since all such points are algebraic, we see that X is actually defined over the algebraic numbers.

Abusing notation, we shall continue to denote by j the map $D^n \to \mathbb{A}^n(\mathbb{C})$ given by $(\tau_1, \ldots, \tau_n) \mapsto (j(\tau_1), \ldots, j(\tau_n))$. From the above observations, the set $\mathfrak{X} := j^{-1}X(\mathbb{C}) \cap D^n$ is definable in $\mathbb{R}_{an,\exp}$ and the restriction of j to $\mathfrak{X}$ induces a bijection between the quadratic imaginary points on $\mathfrak{X}$ and the special points in $X(\mathbb{C})$.

In order to use Theorem 4.5 to estimate the size of the set of rational points on $\mathfrak{X}$, and, hence, the set of special points on $X(\mathbb{C})$, we need to identify $\mathfrak{X}^{\mathrm{alg}}$. Amusingly, in the course of the determination of $\mathfrak{X}^{\mathrm{alg}}$ the Pila-Wilkie bounds are applied to another definable set.

Proposition 5.1. — *The image of $\mathfrak{X}^{\mathrm{alg}}$ in $X(\mathbb{C})$ is a finite union of varieties which, up to permutation of the coordinates, have the form $S \times V$ where S is a special subvariety of $\mathbb{A}^m_{\mathbb{C}}$ of dimension at least one and V is a subvariety of $\mathbb{A}^{n-m}$ for some $m \leq n$.*

Proof (sketch) — One observes first that if $A \subseteq \mathfrak{X}$ is any set and Y is the Zariski closure of A in $(\mathbb{P}^1_{\mathbb{C}})^n$, then $Y(\mathbb{C}) \cap \mathfrak{h}^n \subseteq j^{-1}X(\mathbb{C})$. Thus, to analyze $\mathfrak{X}^{\mathrm{alg}}$ we may restrict attention to sets of the form $Y(\mathbb{C}) \cap D^n$ where Y is an algebraic variety with $Y(\mathbb{C}) \cap \mathfrak{h}^n \subseteq j^{-1}X(\mathbb{C})$. In particular, it follows that $\mathfrak{X}^{\mathrm{alg}}$ may be expressed as a finite union of sets of the form $Y(\mathbb{C}) \cap D^n$ where Y is an irreducible, positive dimensional algebraic variety for which $\dim(Y(\mathbb{C}) \cap D^n) = \dim(Y)$, $Y(\mathbb{C}) \cap \mathfrak{h}^n \subseteq j^{-1}X(\mathbb{C})$, and Y is maximal with respect to inclusion amongst varieties with these properties. For such a variety Y, we consider the following definable set

$$S_Y := \{\gamma \in (\mathrm{PSL}_2(\mathbb{R}))^n \ : \ \dim((\gamma \cdot Y)(\mathbb{C}) \cap \mathfrak{X}) = \dim(Y)\}.$$

Using the fact that $(\gamma \cdot Y)(\mathbb{C}) \cap \mathfrak{h}^n \subseteq j^{-1}X(\mathbb{C})$ for every $\gamma \in (\mathrm{PSL}_2(\mathbb{Z}))^n$ one shows that S_Y contains many integral points where “many” means more than the Pila-Wilkie bounds would permit if S_Y^{alg} were empty. For any semi-algebraic subset $I \subseteq S_Y$, the set $\bigcup_{\gamma \in I} (\gamma \cdot Y)(\mathbb{C}) \cap \mathfrak{X}$ is a semi-algebraic subset of $\mathfrak{X}$ containing Y. Hence, by maximality of Y, Y is stabilized by many infinite connected semi-algebraic subsets of S_Y from which one may deduce that Y is covered by homogeneous spaces. $\square$

Remark 5.2. — Proposition 5.1 may be regarded as a functional modular analogue of the Lindemann-Weierstrass Theorem. That is, we may rephrase the conclusion of the proposition as follows. For each $i \leq n$ let $f_i : (0,1) \to \mathfrak{h}$ be a nonconstant, real analytic semi-algebraic function. If the functions $j(f_1(t)), \ldots, j(f_n(t))$ are algebraically dependent over $\mathbb{Q}$, then there is some $\gamma \in \mathrm{PSL}_2(\mathbb{Q})$ and $i < j \leq n$ for which the functional equation $f_i(t) = \gamma \cdot f_j(t)$ holds.

Our countervailing inequalities come from Siegel's theorem on the growth of the class number [44]. First a definition.

Definition 5.3. — *Suppose that $a, b, c \in \mathbb{Z}$ are integers without a common factor and that $\tau \in \mathbb{C}$ is a complex number satisfying $a\tau^2 + b\tau + c = 0$. We define the discriminant* *of τ to be $\Delta(\tau) := |b^2 - 4ac|$. For an n-tuple $\tau = (\tau_1, \ldots, \tau_n)$ of quadratic numbers we define $\Delta(\tau)$ to be* $\max\{\Delta(\tau_i) \ : \ i \leq n\}$.

Theorem 5.4. — *For any positive number $\epsilon > 0$ there is a constant $C' = C'(\epsilon)$ so that for any quadratic imaginary number $\tau \in \mathfrak{h}$ we have $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \geq C'\Delta(\tau)^{\frac{1}{2}-\epsilon}$.*

Proof (sketch) — From the theory of complex multiplication one knows that $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(\Delta(\tau))$ while Siegel's theorem [45] gives the estimate $h(\Delta(\tau)) \geq C'\Delta(\tau)^{\frac{1}{2}-\epsilon}$. □

With these results in place we may finish sketching the proof of Theorem 1.1.

As we observed above, X is defined over some number field K. Arguing by induction, Proposition 5.1 shows that $\mathfrak{X}^{\mathrm{tr}}$ contains infinitely many quadratic imaginary points, and, hence, such points with arbitrarily large discriminant. If $a \in \mathfrak{X}^{\mathrm{tr}}$ is a quadratic imaginary point, then for each $\sigma \in \mathrm{Gal}(K^{\mathrm{alg}}/K)$ there is some $a' \in \mathfrak{X}^{\mathrm{tr}}$ with $j(a') = \sigma(j(a))$ and $\Delta(a) = \Delta(a')$. Hence, for arbitrarily large t we would have at least $(C'(\frac{1}{6})/[K : \mathbb{Q}])t^{\frac{1}{3}}$ quadratic imaginary points in $\mathfrak{X}^{\mathrm{tr}}$ of discriminant t. On the other hand, it follows from Theorem 4.5 that for some C independent of t we have fewer than $Ct^{\frac{1}{4}}$ such points. For t large enough, these conditions are inconsistent with each other. □

6. FURTHER RESULTS

As mentioned in the introduction, this method of proof has been employed successfully for several other theorems in diophantine geometry. Pila and Zannier [38] reproved the Manin-Mumford conjecture (Raynaud's theorem [41]) that if A is an abelian variety over the complex numbers and $X \subseteq A$ is a closed subvariety, then the intersection of $X(\mathbb{C})$ with the torsion subgroup of $A(\mathbb{C})$ is a finite union of cosets. Peterzil and Starchenko showed in [31] how to extend the arguments from [38] to semi-abelian varieties. Using this method, Masser and Zannier proved the first (and to date only) non-split case of Pink's generalization of the André-Oort conjecture to mixed Shimura varieties in [24, 25, 26]. They consider the Legendre family of elliptic curves $\{E_\lambda\}_{\lambda \in \mathbb{C} \smallsetminus \{0,1\}}$ where the affine equation for E_λ is $y^2 = x(x-1)(x-\lambda)$ and show that if $P, Q \in E_\lambda(\mathbb{C}(\lambda))$ are two $\mathbb{Z}$-linearly independent points on the generic fibre of this family, then there are only finitely many values of λ for which both P and Q specialize to torsion points. It bears noting that [38] and [25] preceded and directly contributed to the ideas used in [36].

The recent work of Habegger and Pila [18] takes this method in a somewhat different direction in that it is proven that if $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is an irreducible curve satisfying an additional technical condition for which X is not contained in a special variety and if we let $\mathcal{S}$ be the union over all special subvarieties $Y \subseteq \mathbb{A}^n_{\mathbb{C}}$ of codimension at least two of $Y(\mathbb{C})$, then $X(\mathbb{C}) \cap \mathcal{S}$ is finite.

One might hope to approach the André-Oort conjecture for higher dimensional Shimura varieties with these techniques. In the case that the ambient Shimura variety is a moduli space for abelian varieties, Peterzil and Starchenko have shown that the requisite theta functions, suitably restricted, are definable in $\mathbb{R}_{an,\exp}$ [32]. The analogue of the Lindemann-Weierstrass theorem is not known, but it appears to be within reach. Recent work of Tsimerman establishes polynomial lower bounds on the Galois orbits of special points in Siegel moduli spaces up to dimension five [48] and recent work of Ullmo and Yafaev gives similar lower bounds in some cases unconditionally and in general under GRH [50].

Thus, it is clear that this line of research has not yet run its course and o-minimal counting arguments will take their place as a powerful tool in diophantine geometry.

REFERENCES

[1] Y. ANDRÉ – *G-functions and geometry*, Aspects of Mathematics, E13. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[2] Y. ANDRÉ – *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire*, J. reine angew. Math. 505 (1998), 203–208.

[3] E. BOMBIERI and J. PILA – *The number of integral points on arcs and ovals*, Duke Math. J. 59 (1989), 337–357.

[4] E. BOMBIERI – *The Mordell conjecture revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 17 (1990), 615–640.

[5] L. BUTLER – *Some cases of Wilkie's conjecture*, preprint, 2011.

[6] C. DAW and A. YAFAEV – *An unconditional proof of the André-Oort conjecture for Hilbert modular surfaces*, Manuscripta Mathematica, `DOI:10.1007/s00229-011-0445-x`, 2011.

[7] J. DENEF and L. VAN DEN DRIES – *p-adic and real subanalytic sets*, Ann. of Math. 128 (1988), 79–138.

[8] L. VAN DEN DRIES – *Remarks on Tarski's problem concerning* $(\mathbf{R}, +, \cdot, \exp)$, Logic colloquium '82 (Florence, 1982), 97–121, Stud. Logic Found. Math. 112, North-Holland, Amsterdam, 1984.

[9] L. VAN DEN DRIES – *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results*, Bull. Amer. Math. Soc. (N.S.) 15 (1986), 189–193.

[10] L. VAN DEN DRIES – *Tame topology and o-minimal structures*, volume 248 of London Math. Soc. Lecture Note Series, Cambridge Univ. Press, Cambridge, 1998.

[11] L. VAN DEN DRIES, A. MACINTYRE, and D. MARKER – *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. 140 (1994), 183–205.

[12] L. VAN DEN DRIES and C. MILLER – *On the real exponential field with restricted analytic functions*, Israel J. Math. 85 (1994), 19–56.

[13] B. EDIXHOVEN – *Special points on the product of two modular curves*, Compositio Math. 114 (1998), 315–328.

[14] B. EDIXHOVEN – *Special points on products of modular curves*, Duke Math. J. 126 (2005), 325–348.

[15] B. EDIXHOVEN and A. YAFAEV – *Subvarieties of Shimura varieties*, Ann. of Math. 157 (2003), 621–645.

[16] A. GABRIELOV – *Projections of semi-analytic sets*, Functional Anal. Appl. 2 (1968), 282–291.

[17] M. GROMOV – *Entropy, homology and semi-algebraic geometry (after Y. Yomdin)*, Séminaire Bourbaki (1985/86), exp. n° 663, Astérisque 145-146 (1987), 225–240.

[18] P. HABEGGER and J. PILA – *Some unlikely intersections beyond André-Oort*, preprint, 2010.

[19] G. JONES, D. MILLER and M. THOMAS – *Mildness and the density of rational points on certain transcendental curves*, Notre Dame J. Formal Logic 52 (2011), 67–74.

[20] G. JONES and M. THOMAS – *The density of algebraic points on certain pfaffian surfaces*, preprint, 2010.

[21] A. G. KHOVANSKIĬ – *Fewnomials*, volume 88 of Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1991.

[22] B. KLINGLER and A. YAFAEV, *The André-Oort conjecture*, preprint, 2006.

[23] J. KNIGHT, A. PILLAY, and C. STEINHORN – *Definable sets in ordered structures II*, Trans. Amer. Math. Soc. 295 (1986), 593–605.

[24] D. MASSER and U. ZANNIER, – *Torsion anomalous points and families of elliptic curves*, C. R. Math. Acad. Sci. Paris 346 (2008), 491–494.

[25] D. MASSER and U. ZANNIER – *Torsion anomalous points and families of elliptic curves*, Amer. J. Math. 132 (2010), 1677–1691.

[26] D. MASSER and U. ZANNIER – Torsion points on families of squares of elliptic curves, Math. Annalen, Online First, 16 February 2011.

[27] B. MOONEN – *Linearity properties of Shimura varieties II*, Compositio Math. 114 (1998), 3–35.

[28] R. NOOT – Correspondances de Hecke, action de Galois et la conjecture d'André-Oort (d'après Edixhoven et Yafaev), Séminaire Bourbaki (2004/05), exp. n° 942, Astérisque 307 (2006), 165–197.

[29] F. OORT – *Some questions in algebraic geometry*, 1995, `http://www.math.uu.nl/people/oort/`

[30] Y. PETERZIL and S. STARCHENKO – *Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one*, Selecta Math. (N.S.) 10 (2004), 525–550.

[31] Y. PETERZIL and S. STARCHENKO – *Around Pila-Zannier: the semiabelian case*, preprint, 2009.

[32] Y. PETERZIL and S. STARCHENKO – *Definability of restricted theta functions and families of abelian varieties*, preprint, 2010.

[33] J. PILA – *Integer points on the dilation of a subanalytic surface*, Q. J. Math. 55 (2004), 207–223.

[34] J. PILA – *Counting rational points on a certain exponential-algebraic surface*, Ann. Inst. Fourier (Grenoble) 60 (2010), 489–514.

[35] J. PILA – *On the algebraic points of a definable set*, Selecta Math. (N.S.) 15 (2009), 151–170.

[36] J. PILA – *O-minimality and the André-Oort conjecture for C^n*, Ann. of Math. 173, (2011), 1779–1840.

[37] J. PILA and A. WILKIE – *The rational points of a definable set*, Duke Math. J. 133 (2006), 591–616.

[38] J. PILA and U. ZANNER – *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (2008), 149–162.

[39] A. PILLAY and C. STEINHORN – *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. 295 (1986), 565–592.

[40] A. PILLAY and C. STEINHORN – *Definable sets in ordered structures. III*, Trans. Amer. Math. Soc. 309 (1988), 469–476.

[41] M. RAYNAUD – *Sous-variétés d'une variété abélienne et points de torsion*, in Arithmetic and geometry, Vol. I, volume 35 of Progr. Math., 327–352, Birkhäuser Boston, Boston, MA, 1983.

[42] J-P. SERRE – *Cours d'arithmétique*, Presses Universitaires de France, 1970.

[43] T. SCANLON – *Counting special points: logic, diophantine geometry and transcendence theory*, Current Events Bulletin, 8 January 2011.

[44] C. SIEGEL – *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. (1) 1935, 83–86.

[45] J. SILVERMAN – *The arithmetic of elliptic curves*, Graduate Texts in Math. 106, Springer-Verlag, New York, 1992.

[46] P. SPEISSEGGER – *The Pfaffian closure of an o-minimal structure*, J. reine angew. Math. 508 (1999), 189–211.

[47] A. TARSKI – *A Decision Method for Elementary Algebra and Geometry*, RAND Corporation, Santa Monica, Calif., 1948.

[48] J. TSIMERMAN – *Brauer-Siegel for arithmetic tori and lower bounds for Galois orbits of special points*, preprint, 2011.

[49] E. ULLMO and A. YAFAEV – *The André-Oort conjecture for products of modular curves*, Arithmetic geometry 431–439, Clay Math. Proc. 8, Amer. Math. Soc., Providence, RI, 2009.

[50] E. ULLMO and A. YAFAEV – *Nombre de classes des tores de multiplication complexe et bornes inférieures pour orbites galoisiennes de points spéciaux*, preprint, 2011.

[51] A. WILKIE – *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. 9 (1996), 1051–1094.

[52] A. WILKIE – *A theorem of the complement and some new o-minimal structures*, Selecta Math. (N.S.) 5 (1999), 397–421.

[53] A. WILKIE – *O-minimal structures*, Séminaire Bourbaki (2007/08), exp. n° 985, Astérisque 326 (2009), 131–142.

[54] Y. YOMDIN – *Volume growth and entropy*, Israel J. Math. 57 (1987), 285–300.

[55] Y. YOMDIN – *C^k-resolution of semialgebraic mappings. Addendum to: "Volume growth and entropy"*, Israel J. Math. 57 (1987), 301–317.

Thomas SCANLON

University of California
Department of Mathematics
Evans Hall
Berkeley, CA 94720-3840 – U.S.A.
E-mail : `scanlon@math.berkeley.edu`