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THE HODGE THEORY OF THE DECOMPOSITION THEOREM [after M. A. de Cataldo and L. Migliorini]

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INTRODUCTION

The Decomposition Theorem is a beautiful theorem about algebraic maps. In the words of MacPherson [Mac83], "it contains as special cases the deepest homological properties of algebraic maps that we know." Since its proof in 1981 it has found spectacular applications in number theory, representation theory and combinatorics. Like its cousin the Hard Lefschetz Theorem, proofs appealing to the Decomposition Theorem are usually difficult to obtain via other means. This leads one to regard the Decomposition Theorem as a deep statement lying at the heart of diverse problems.

The Decomposition Theorem was first proved by Beilinson, Bernstein, Deligne and Gabber [BBD]. Their proof proceeds by reduction to positive characteristic in order to use the Frobenius endomorphism and its weights, and ultimately rests on Deligne's proof of the Weil conjectures. Some years later Saito obtained another proof of the Decomposition Theorem as a corollary of his theory of mixed Hodge modules [Sai87, Sai89]. Again the key is a notion of weight.

More recently, de Cataldo and Migliorini discovered a simpler proof of the Decomposition Theorem [dCM02, dCM05]. The proof is an ingenious reduction to statements about the cohomology of smooth projective varieties, which they establish via Hodge theory. In their proof they uncover several remarkable geometric statements which go a long way to explaining "why" the Decomposition Theorem holds, purely in the context of the topology of algebraic varieties. For example, their approach proves that the intersection cohomology of a projective variety is of a motivic nature ("André motivated") [dCM15]. Their techniques were adapted by Elias and the author to prove the existence of Hodge theories attached to Coxeter systems ("Soergel modules"), thus proving the Kazhdan-Lusztig positivity conjecture [EW14].

The goal of this article is to provide an overview of the main ideas involved in de Cataldo and Migliorini's proof. A striking aspect of the proof is that it gathers the Decomposition Theorem together with several other statements generalising the Hard Lefschetz Theorem and the Hodge-Riemann Bilinear Relations (the "Decomposition Theorem Package"). Each ingredient is indispensable in the induction. One is left with the impression that the Decomposition Theorem is not a theorem by itself, but rather belongs to a family of statements, each of which sustains the others.

Before stating the Decomposition Theorem we recall two earlier theorems concerning the topology of algebraic maps. The first (Deligne's Degeneration Theorem) is an instance of the Decomposition Theorem. The second (Grauert's Theorem) provides an illustration of the appearance of a definite form, which eventually forms part of the "Decomposition Theorem Package".

0.1. Deligne's Degeneration Theorem

Let $f : X \to Y$ be a smooth (i.e. submersive) projective morphism of complex algebraic varieties. Deligne's theorem asserts that the Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(X, \mathbb{Q}) \tag{1}$$

is degenerate (i.e. $E_2 = E_\infty$). Of course such a statement is false for submersions between manifolds (e.g. the Hopf fibration). The theorem asserts that something very special happens for smooth algebraic maps.

Let us recall how one may construct the Leray spectral sequence. In order to compute the cohomology of X we replace the constant sheaf $\mathbb{Q}_X$ on X by an injective resolution. Its direct image on Y then has a natural "truncation" filtration whose successive subquotients are the (shifted) higher direct image sheaves $R^q f_* \mathbb{Q}_X[-q]$. This filtered complex of sheaves gives rise to the Leray spectral sequence.

In fact, Deligne proved that there exists a decomposition in the derived category of sheaves on Y:

$$Rf_* \mathbb{Q}_X \cong \bigoplus_{q \geq 0} R^q f_* \mathbb{Q}_X[-q] \tag{2}$$

(i.e. the filtration of the previous paragraph splits). The decomposition in (2) implies the degeneration of (1), and in fact is the universal explanation for such a degeneration. Deligne also proved that each local system $R^q f_* \mathbb{Q}_X$ is semi-simple. Hence the object $Rf_* \mathbb{Q}_X$ is as semi-simple as we could possibly hope. This is the essence of the Decomposition Theorem, as we will see.

Because $f : X \to Y$ is smooth and projective any fibre of f is a smooth projective variety. Deligne deduces the decomposition in (2) by applying the Hard Lefschetz Theorem to the cohomology of the fibres of f. Thus the decomposition of $Rf_* \mathbb{Q}_X$ is deduced from a deep fact about the global cohomology of a smooth projective variety. This idea occurs repeatedly in the proof of de Cataldo and Migliorini.

0.2. Grauert's Theorem

Let X denote a smooth projective surface and let $C = \bigcup_{i=1}^m C_i$ denote a connected union of irreducible curves on X. It is natural to ask whether C can be contracted. That is, whether there exists a map

$$f : X \to Y$$

which is an isomorphism on the complement of C and contracts C to a point. Of course such a map of topological spaces always exists, but it is a subtle question if one requires f and Y to be algebraic or analytic. An answer is given by Grauert's theorem: f exists analytically if and only if the intersection form

$$([C_i] \cap [C_j])_{1 \leq i,j \leq k} \tag{3}$$

is negative definite. For example, if C is irreducible (i.e. $k = 1$) then C can be contracted if and only if C has negative self-intersection in X.

Let us assume that such an f exists, and let $y \in Y$ denote the image of C. Then in this case the Decomposition Theorem asserts a decomposition in the derived category of sheaves on Y

$$Rf_*\mathbb{Q}_X[2] = IC(Y) \oplus \bigoplus_{i=1}^{k} \mathbb{Q}_y \tag{4}$$

where $IC(Y)$ is a complex of sheaves on Y which is a simple object in the category of perverse sheaves. Again, (4) can be interpreted in the language of perverse sheaves as saying that the object $Rf_*\mathbb{Q}_X[2]$ is as semi-simple as possible.

Remarkably, the decomposition in (4) is equivalent to the fact that the intersection form in (3) is non-degenerate. Thus in this example the Decomposition Theorem is a consequence of a topological fact about contractibility of curves on a surface. Note also that here the geometric theorem that we are using (negative definiteness) is stronger than what we need for the Decomposition Theorem (non-degeneracy). As we will see, keeping track of such signs plays an important role in de Cataldo and Migliorini's proof.

0.3. Structure of the paper

This paper consists of three sections. In §1 we recall the necessary background from topology, Hodge theory and perverse sheaf theory and state the Decomposition Theorem. In §2 we discuss de Cataldo and Migliorini's proof for semi-small maps. The case of semi-small maps has the advantage of illustrating several of the general features of the proof very well, whilst being much simpler in structure. In §3 we give the statements and an outline of the main steps of the induction establishing the theorem for arbitrary maps.

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1. BACKGROUND

In this section we briefly recall the tools (intersection forms, classical Hodge theory, perverse sheaves) which we will be using throughout this paper. We discuss the relationship between perverse sheaves and the weak Lefschetz theorem and state the Decomposition Theorem.

Remark 1.1. — A remark on coefficients: The natural setting for the Decomposition Theorem and its relatives is that of sheaves of $\mathbb{Q}$-vector spaces. However, at some points below it is necessary to consider sheaves of $\mathbb{R}$-vector spaces (usually due to limit arguments). To avoid repeated change of coefficients we have chosen to work with $\mathbb{R}$-coefficients throughout. All of the arguments of this paper are easily adapted for $\mathbb{Q}$-coefficients, as the reader may readily check.

1.1. Algebraic Topology

All spaces will be complex algebraic varieties equipped with their classical (metric) topology. The dimension of a complex algebraic variety will always mean its complex dimension. We do not assume that varieties are irreducible, and dimension means the supremum over the dimension of its components. Given a variety Z we denote by

$$H^*(Z) = H^*(Z, \mathbb{R}) \quad \text{and} \quad H_*(Z) = H_*(Z, \mathbb{R})$$

its singular cohomology and singular homology with closed supports ("Borel-Moore homology"), with coefficients in the real numbers.

Any irreducible subvariety $Z' \subset Z$ of dimension p has a fundamental class

$$[Z'] \in H_{2p}(Z).$$

If Z is of dimension n then $H_{2n}(Z)$ has a basis given by the fundamental classes of irreducible components of maximal dimension.

If X is smooth of dimension n then (after choosing once and for all an orientation of $\mathbb{C}$) Poincaré duality gives a canonical isomorphism

$$H_p(X) \xrightarrow{\sim} H^{2n-p}(X). \tag{5}$$

If X is in addition compact then $H^*(X)$ has a non-degenerate *Poincaré form*

$$(-,-) : H^{2n-p}(X) \times H^p(X) \to \mathbb{R}$$

and $H_*(X)$ is equipped with a non-degenerate *intersection form*

$$\cap : H_p(X) \times H_q(X) \to H_{p+q-2n}(X).$$

These forms match under Poincaré duality. If X is smooth we will often identify $H^*(X)$ with the real de Rham cohomology of X. In de Rham cohomology the Poincaré form is given by the integral

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$

Suppose Z is a proper closed subvariety inside a smooth n-dimensional variety X. If $p + q = 2n$ the inclusion $Z \hookrightarrow X$ gives rise to an intersection form (see e.g. [Ful84, Chapter 19])

$$H_p(Z) \times H_q(Z) \to \mathbb{R}.$$

Geometrically this corresponds to moving cycles on Z into X until they become transverse, and then counting the number of intersection points. If X is proper and connected the map $H_*(Z) \to H_*(X)$ is an isometry for intersection forms.

1.2. Hodge Theory

Let X be a smooth and connected projective variety of complex dimension n. Let $H^*(X)$ denote the de Rham cohomology of X with coefficients in the real numbers. Throughout it will be convenient to shift indices; consider the finite-dimensional graded vector space

$$H = \bigoplus_{i \in \mathbb{Z}} H^i \quad \text{where} \quad H^i := H^{n+i}(X).$$

Under this normalization the Poincaré pairing induces canonical isomorphisms

$$H^{-i} \xrightarrow{\sim} (H^i)^\vee \quad \text{for all } i \in \mathbb{Z} \tag{6}$$

where $(H^i)^\vee$ denotes the dual vector space.

Theorem 1.2 (The Hard Lefschetz Theorem). — *Let $\omega \in H^2(X)$ denote the Chern class of an ample line bundle. For all $i \geq 0$, multiplication by ω^i induces an isomorphism*

$$\omega^i : H^{-i} \xrightarrow{\sim} H^i. \tag{7}$$

Let $P^{-i} \subset H^{-i}$ denote the primitive subspace:

$$P^{-i} := \ker(\omega^{i+1} : H^{-i} \to H^{i+2}).$$

The Hard Lefschetz Theorem gives the primitive decomposition:

$$\bigoplus_{i \geq 0} \mathbb{R}[\omega]/(\omega^{i+1}) \otimes_{\mathbb{R}} P^{-i} \xrightarrow{\sim} H.$$

Remark 1.3. — Consider the Lie algebra $\mathfrak{sl}_2 := \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ with

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Hard Lefschetz Theorem is equivalent to the existence of a $\mathfrak{sl}_2$-action on H with $e(x) = \omega \wedge x$ and $h(x) = jx$ for all $x \in H^j$. The primitive decomposition is the isotypic decomposition and the primitive subspaces are the lowest weight spaces.

We now state the Hodge-Riemann bilinear relations, for which we need a little more notation. For $i \geq 0$ the form

$$Q(\alpha, \beta) := \int \omega^i \wedge \alpha \wedge \beta$$

on H^{-i} is symmetric if $n-i$ is even and alternating if $n-i$ is odd. It is non-degenerate by the Hard Lefschetz theorem. Given a real vector space V we denote by $V_{\mathbb{C}}$ its complexification. The form

$$\kappa(\alpha, \beta) := (\sqrt{-1})^{n-i} Q(\alpha, \overline{\beta})$$

on $H^{-i}_{\mathbb{C}}$ is Hermitian and non-degenerate.

Consider the Hodge decomposition and corresponding primitive spaces

$$H^j_{\mathbb{C}} = \bigoplus_{p+q=n+j} H^{p,q}, \qquad P^{p,q} := P^{p+q-n}_{\mathbb{C}} \cap H^{p,q}.$$

THEOREM 1.4 (Hodge-Riemann bilinear relations). — *The Hodge decomposition is orthogonal with respect to κ. Moreover, if $\alpha \in P^{p,q}$ is non-zero and $k := p+q$ then*

$$(\sqrt{-1})^{p-q-k}(-1)^{k(k-1)/2}\kappa(\alpha,\overline{\alpha}) > 0.$$

Remark 1.5. — The Hodge-Riemann relations imply that the restriction of the Hermitian form κ to $P^{p,q}$ is definite of a fixed sign. This fact is crucial below. As long as the reader keeps this definiteness in mind, the precise nature of the signs can be ignored on a first reading.

Remark 1.6. — More generally, hard Lefschetz and the Hodge-Riemann bilinear relations are valid for any class $\omega \in H^2(X)$ in the ample cone (the convex hull of all strictly positive real multiples of ample classes).

A *(real, pure) Hodge structure* of weight k is a finite-dimensional real vector space V together with a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Hodge structures form an abelian category in a natural way. A *polarisation* of a real Hodge structure of weight k is a bilinear form Q on V which is symmetric if k is even, anti-symmetric if k is odd and such that the corresponding Hermitian form $\kappa(\alpha,\beta) := (\sqrt{-1})^k Q(\alpha,\overline{\beta})$ on $V_{\mathbb{C}}$ satisifies the Hodge-Riemann Bilinear Relations (Theorem 1.4). For example, for $i \geq 0$ each H^{-i} above is a Hodge structure of weight $n-i$ and $P^{-i} \subset H^{-i}$ is a Hodge substructure polarised by Q.

1.3. Constructible and perverse sheaves

In the following we recall the formalism of the constructible derived category. For more detail the reader is referred to [dCM09, §5] and the references therein.

We denote by $D^b_c(Y)$ the constructible derived category of sheaves of $\mathbb{R}$-vector spaces on Y. This is a triangulated category with shift functor [1]. Given an object $\mathcal{F} \in D^b_c(Y)$ we denote by $\mathcal{H}^i(\mathcal{F})$ its cohomology sheaves. Given a morphism $f : X \to Y$ of algebraic varieties we have functors

$$\begin{tikzcd} D^b_c(X) \arrow[r, bend left, "{f_*, f_!}"] & D^b_c(Y) \arrow[l, bend left, "{f^*, f^!}"] \end{tikzcd}$$

(we only consider derived functors and write f_* instead of Rf_*, etc.). Verdier duality is denoted $\mathbb{D} : D^b_c(Y) \to D^b_c(Y)$.

We let $\mathbb{R}_Z$ and ω_Z denote the constant and dualizing sheaves on Z. If Y is smooth we have $\omega_Y = \mathbb{R}_Y[2\dim Y]$ canonically (Poincaré duality). Given $\mathcal{F} \in D^b_c(Y)$ we denote its hypercohomology by $H(Y,\mathcal{F})$. In the notation of § 1.1 we have

$$H^j(Y) = H^j(Y,\mathbb{R}_Y) \quad \text{and} \quad H_j(Y) = H^{-j}(Y,\omega_Y).$$

The full subcategories

$$
\begin{aligned}
{}^{p}D^{\leq 0}(Y) &:= \{\mathcal{F} \in D^b_c(Y) \mid \dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i \text{ for all } i\},\\
{}^{p}D^{\geq 0}(Y) &:= \{\mathcal{F} \in D^b_c(Y) \mid \dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}\mathcal{F}) \leq -i \text{ for all } i\}
\end{aligned}
$$

define a t-structure on $D^b_c(Y)$ whose heart is the abelian category $P_Y \subset D^b_c(Y)$ of perverse sheaves (for the middle perversity). (The standard warning that perverse sheaves are not sheaves, but rather complexes of sheaves is repeated here.)

Define ${}^{p}D^{\leq m} := {}^{p}D^{\leq 0}[-m]$ and ${}^{p}D^{\geq m} := {}^{p}D^{\geq 0}[-m]$. We denote by ${}^{p}\tau_{\leq m}$ and ${}^{p}\tau_{\geq m}$ the perverse truncation functors

$$
{}^{p}\tau_{\leq m} : D^b_c(Y) \to {}^{p}D^{\leq m}(Y) \quad \text{and} \quad {}^{p}\tau_{\geq m} : D^b_c(Y) \to {}^{p}D^{\geq m}(Y)
$$

which are right (resp. left) adjoint to the inclusion functors. Given $\mathcal{F} \in D^b_c(Y)$ its perverse cohomology groups are ${}^{p}\mathcal{H}^i(\mathcal{F}) := {}^{p}\tau_{\leq 0}{}^{p}\tau_{\geq 0}(\mathcal{F}[i]) \in P_Y$.

Remark 1.7. — For fixed $\mathcal{F} \in D^b_c(Y)$ we have ${}^{p}\tau_{\leq i}\mathcal{F} = 0$ for $i \ll 0$ and ${}^{p}\tau_{\geq i}\mathcal{F} = 0$ for $i \gg 0$. It is convenient to view $\mathcal{F}$ as equipped with a canonical exhaustive filtration (in the triangulated sense)

$$
\dots \to {}^{p}\tau_{\leq i}\mathcal{F} \to {}^{p}\tau_{\leq i+1}\mathcal{F} \to \dots
$$

with subquotients the (shifted) perverse sheaves ${}^{p}\mathcal{H}^i(\mathcal{F})[-i]$.

Given any locally closed, smooth and connected subvariety $Z \subset Y$ and a local system $\mathcal{L}$ of $\mathbb{R}$-vector spaces on Z we denote by $IC(\overline{Z}, \mathcal{L})$ the intersection cohomology complex of $\mathcal{L}$. The object $IC(\overline{Z}, \mathcal{L}) \in P_Y$ is simple if $\mathcal{L}$ is, and all simple perverse sheaves are of this form. For example, if $\overline{Z}$ is smooth and $\mathcal{L}$ extends as a local system $\overline{\mathcal{L}}$ to $\overline{Z}$ then $IC(\overline{Z}, \mathcal{L}) = \overline{\mathcal{L}}[\dim Z]$. We write $IH(\overline{Z}, \mathcal{L}) = H(Y, IC(\overline{Z}, \mathcal{L}))$ for the intersection cohomology of $\overline{Z}$ with coefficients in $\mathcal{L}$. If $\mathcal{L}$ is the trivial local system we write $IC(\overline{Z})$ and $IH(\overline{Z})$ instead of $IC(\overline{Z}, \mathcal{L})$ and $IH(\overline{Z}, \mathcal{L})$.

Let us fix a Whitney stratification $Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$ and denote by $i_\mu : Y_\mu \hookrightarrow Y$ the inclusion. If we fix a stratum $Y_\lambda \subset Y$ and a local system $\mathcal{L}$ on Y_λ then $IC(\overline{Y_\lambda}, \mathcal{L})$ is uniquely characterised by the conditions:

$$
i_\lambda^* IC(\overline{Y}_\lambda, \mathcal{L}) = \mathcal{L}[\dim Z], \tag{8}
$$

$$
\mathcal{H}^j(i_\mu^* IC(\overline{Y}_\lambda, \mathcal{L})) = 0 \quad \text{for } j \geq -\dim Y_\mu \text{ and } \mu \neq \lambda, \tag{9}
$$

$$
\mathcal{H}^j(i_\mu^! IC(\overline{Y}_\lambda, \mathcal{L})) = 0 \quad \text{for } j \leq -\dim Y_\mu \text{ and } \mu \neq \lambda. \tag{10}
$$

At several points in de Cataldo and Migliorini's proof vanishing theorems for perverse sheaves on affine varieties play an important role. Recall Artin-Grothendieck vanishing (see e.g. [Laz04, 3.1.13]): if $\mathcal{F}$ is a constructible *sheaf* (i.e. $\mathcal{F} = \mathcal{H}^0(\mathcal{F})$) on an affine variety U then

$$
H^j(U, \mathcal{F}) = 0 \quad \text{for } j > \dim U. \tag{11}
$$

The following proposition characterises the perverse sheaves as those complexes for which such vanishing is universal:

Proposition 1.8. — *$\mathcal{F} \in D^b_c(Y)$ belongs to ${}^pD^{\leq 0}(Y)$ if and only if, for all affine open subvarieties $U \subset Y$, we have*

$$H^j(U, \mathcal{F}) = 0 \quad \textit{for } j > 0.$$

Similarly, $\mathcal{F} \in {}^pD^{\geq 0}$ if and only if for all affine open U we have

$$H^j_c(U, \mathcal{F}) = 0 \quad \textit{for } j < 0,$$

where $H^j_c(U, \mathcal{F})$ denotes cohomology with compact supports.

Proof (Sketch) — The first statement implies the second, by Verdier duality. The implication $\Rightarrow$ is easily deduced from the definition of the perverse t-structure and Artin-Grothendieck vanishing (11). For the implication $\Leftarrow$ see [BBD, 4.1.6].

Now suppose that Y is projective and let $i : D \hookrightarrow Y$ denote the inclusion of a hyperplane section and $j : Y \setminus D \hookrightarrow Y$ the open inclusion of its (affine) complement. After taking cohomology of the distinguished triangle $j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{[1]}$ or its dual and applying the above vanishing we deduce:

Theorem 1.9 (Weak Lefschetz for Perverse Sheaves). — *Let $\mathcal{F} \in D^b_c(Y)$ be perverse.*

- *The restriction map $H^j(Y, \mathcal{F}) \to H^j(D, i^*\mathcal{F})$ is an isomorphism for $j < -1$ and is injective for $j = -1$.*
- *The pushforward map $H^j(D, i^!\mathcal{F}) \to H^j(Y, \mathcal{F})$ is an isomorphism for $j > 1$ and is surjective for $j = 1$.*

1.4. The Decomposition Theorem

Definition 1.10. — *An object in $D^b_c(Y)$ is* semi-simple *if it is isomorphic to a direct sum of shifts of intersection cohomology complexes of semi-simple local systems.*

Theorem 1.11 (Decomposition Theorem). — *If $f : X \to Y$ is projective and X is smooth then $f_* \mathbb{R}_X$ is semi-simple.*

Remark 1.12. — Some remarks concerning the generality of the Decomposition Theorem discussed below:

- One could drop the assumption that X be smooth and replace $f_*\mathbb{R}_X$ by $f_*IC(X)$. This formulation follows from the above via resolution of singularities. By Chow's lemma we could also replace "f projective" by "f proper". The formulation above is preferred because it is the one addressed in this paper.
- In Saito's theory the Decomposition Theorem is proved more generally for $f_*IC(X, \mathcal{L})$ where $\mathcal{L}$ is any local system underlying a polarisable variation of Hodge structure on a Zariski open subvariety of X. It is likely that the techniques discussed here could handle this case (after reducing to the normal crossing situation and using the existence of a pure Hodge structure on $IH(X, \mathcal{L})$ established by Kashiwara-Kawai [KK87], and Cattani-Kaplan-Schmid [CKS87]). Recently

El Zein, Lê and Ye have proposed another proof of the Decomposition Theorem in this level of generality [EY14, EL14, EL15].
– More general still are the results of Sabbah [Sab05] and Mochizuki [Moc07] which establish the semi-simplicity of $f_* IC(X, \mathcal{L})$ where $\mathcal{L}$ is *any* semi-simple $\mathbb{C}$-local system. The proof is via a generalization of Saito's theory, and probably goes far beyond what is possible with the techniques discussed here.

2. SEMI-SMALL MAPS

2.1. The Decomposition Theorem for semi-small maps

Suppose (as we will assume throughout this paper) that X is smooth, connected and projective of complex dimension n and that $f : X \to Y$ is a surjective algebraic map. Throughout we fix a stratification

$$Y = \bigsqcup_{\Lambda} Y_\lambda$$

of Y adapted to f. In particular, each Y_λ is connected and, over each stratum, $f : f^{-1}(Y_\lambda) \to Y_\lambda$ is a topologically locally trivial fibration in (typically singular) varieties.

Definition 2.1. — *The map f is* semi-small *if for all $\lambda \in \Lambda$ and some (equivalently all) $y \in Y_\lambda$ we have*

$$\dim f^{-1}(y) \leq \frac{1}{2}(\dim Y - \dim Y_\lambda). \tag{12}$$

Semi-small maps play an important role in the theory of perverse sheaves. This is mainly because of the following fact (which is a straightforward consequence of the proper base change theorem and the Verdier self-duality of $f_*\mathbb{R}_X[n]$):

Proposition 2.2. — *If f is semi-small then $f_*\mathbb{R}_X[n]$ is perverse.*

Remark 2.3. — From the definition it follows that a semi-small map is finite on any open stratum of Y. It can be useful to think of semi-small maps as being the finite maps of perverse sheaf theory. (Compare with the fact that the (derived) direct image of the constant sheaf along a projective morphism is a sheaf if and only if f is finite.)

Theorem 2.4 (Decomposition Theorem for semi-small maps)
If f is semi-small then $f_*\mathbb{R}_X[n]$ is a semi-simple perverse sheaf. More precisely:
– *We have a canonical decomposition*

$$f_*\mathbb{R}_X[n] = \bigoplus_{\lambda \in \Lambda} IC(\overline{Y}_\lambda, \mathcal{L}_\lambda) \tag{13}$$

where each $\mathcal{L}_\lambda$ is the local system on Y_λ associated to $y \mapsto H^{\dim Y - \dim Y_\lambda}(f^{-1}(y))$.
– *Each local system $\mathcal{L}_\lambda$ is semi-simple.*

Remark 2.5. — The semi-small case is special because the decomposition (13) is canonical and explicit. For general maps the decomposition is not canonical and it is difficult to say a priori which summands occur in the direct image.

Remark 2.6. — An important aspect of the Decomposition Theorem (already non-trivial in Deligne's Degeneration Theorem) is that each local system $\mathcal{L}_\lambda$ is semi-simple. In the semi-small case the representations corresponding to each $\mathcal{L}_\lambda$ are dual to the permutation representation of $\pi_1(Y_\lambda, y)$ on the irreducible components of the fibre $f^{-1}(y)$ of "maximal" (i.e $= \frac{1}{2}(\dim Y - \dim Y_\lambda)$) dimension. In particular, each representation factors over a finite group and semi-simplicity follows from Maschke's Theorem in finite group theory.

Remark 2.7. — The decomposition (13) implies that the cohomology of the fibres of f is completely determined by the local systems $\mathcal{L}_\lambda$ and the singularities of Y. Thus much of the topology of f is determined by the irreducible components of each fibre, and the monodromy along each stratum. This gives a hint as to the nature of the Decomposition Theorem.

Remark 2.8. — The decomposition (13) gives a canonical decomposition of cohomology:

$$H^{*+n}(X) = \bigoplus_{\lambda \in \Lambda} IH^*(\overline{Y}_\lambda, \mathcal{L}_\lambda). \tag{14}$$

In [dCM04] it is shown that this decomposition is motivic (i.e. given by algebraic cycles in $X \times_Y X$). For example if X is proper this gives a canonical decomposition of the Chow motive of X [dCM04, Theorem 2.4.1].

We say that $\omega \in H^2(X)$ is a *semi-small class* if ω is the first Chern class of a line bundle $\mathcal{L}$, some positive power of which is globally generated and whose global sections yield a semi-small map $X \to Y$.

Theorem 2.9 (Hard Lefschetz and Hodge-Riemann for semi-small classes)
Let $\omega \in H^2(X)$ be a semi-small class. Then multiplication by ω satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.

Remark 2.10. — More generally one can show that if $f : X \to Y$ is any morphism, $\mathcal{L}$ is an ample line bundle on Y and ω is the Chern class of $f^*\mathcal{L}$ then ω satisfies hard Lefschetz if and only if f is semi-small, see [dCM02, Proposition 2.2.7].

Remark 2.11. — If one knows that the hypercohomology of each summand appearing in the Decomposition Theorem satisfies hard Lefschetz and the Hodge-Riemann relations (as follows for example from Saito's theory) then Theorem 2.9 is an immediate consequence of Theorem 2.4. A key insight of de Cataldo and Migliorini is to realise that the Decomposition Theorem in the semi-small case is implied by Theorem 2.9, as we will explain below.

Remark 2.12. — Theorem 2.9 can be used to put pure Hodge structures on each summand in (14).

2.2. Local study of the Decomposition Theorem: semi-small case

Suppose that $f : X \to Y$ is as in the previous section with f semi-small. From the definition of a semi-small map it is immediate that the dimension of any fibre of f is at most half of the dimension of X, and that equality can only occur at finitely many points in Y. It is useful to think of these points as the "most singular points" of f.

Example 2.13. — The first interesting example of a semi-small map is that of a contraction of curves on a surface appearing in Grauert's theorem (as discussed in the introduction). The image of the contracted curves is typically a singular point of Y, which is an example of a $y \in Y$ that we study below.

Let us assume that X is of even dimension $n = 2m$. We fix a point $y \in Y$ such that $\dim f^{-1}(y) = m$. Consider the Cartesian diagram:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "f"] & X \arrow[d, "f"] \\ \{y\} \arrow[r, "i"] & Y. \end{tikzcd} \tag{15}$$

The fibre $F = f^{-1}(y)$ is typically reducible. If we denote by $F_1, F_2, \ldots, F_k$ the irreducible components of F of dimension m then we have

$$H_n(F) = \bigoplus_{i=1}^{k} \mathbb{R}[F_i] \tag{16}$$

where $[F_i] \in H_n(F)$ denotes the fundamental class of $F_i \subset F$. Because each F_i is half-dimensional inside X the inclusion $F \hookrightarrow X$ equips $H_n(F)$ with a symmetric intersection form (see § 1.1)

$$H_n(F) \times H_n(F) \to \mathbb{R}. \tag{17}$$

We will call this form the *local intersection form* (at y).

The Decomposition Theorem predicts

$$f_*\mathbb{R}_X[n] = \mathcal{F} \oplus i_*(H^n(F))_y \tag{18}$$

where $i_*(H^n(F))_y$ denotes the constant sheaf on $\{y\}$ with stalk $H^n(F) = H_n(F)^\vee$. (Here $\mathcal{F}$ is some perverse sheaf, whose structure can be ignored for the moment.) We will say that the *Decomposition Theorem holds at* y if the decomposition (18) is valid.

Remark 2.14. — Let us justify this terminology. In de Cataldo and Migliorini's proof one knows by induction that the restriction of $\mathcal{F}$ to the complement of all of the point strata is semi-simple. It is then not difficult to prove that the decomposition (18) for all point strata (or "most singular points") is equivalent to the Decomposition Theorem for $f_*\mathbb{R}_X[n]$. Thus the innocent looking (18) is the key to the Decomposition Theorem for Semi-Small Maps.

How do we decide whether the Decomposition Theorem holds at y? The Decomposition Theorem holds at y if and only if the skyscraper sheaf $i_*\mathbb{R}_y$ occurs with multiplicity equal to the dimension of $H_n(F)$ as a summand of $f_*\mathbb{R}_X[n]$. We can rephrase this as follows: If we consider the pairing

$$\mathrm{Hom}(i_*\mathbb{R}_y, f_*\mathbb{R}_X[n]) \times \mathrm{Hom}(f_*\mathbb{R}_X[n], i_*\mathbb{R}_y) \to \mathrm{End}(i_*\mathbb{R}_y) = \mathbb{R}, \tag{19}$$

then the Decomposition Theorem holds at y if and only if the rank of the pairing (19) is $\dim H_n(F)$.

Lemma 2.15. — *We have canonical isomorphisms*

$$\mathrm{Hom}(i_*\mathbb{R}_y, f_*\mathbb{R}_X[n]) = H_n(F) = \mathrm{Hom}(f_*\mathbb{R}_X[n], i_*\mathbb{R}_y). \tag{20}$$

Proof — By adjunction, the proper base change theorem and the identification $\mathbb{R}_X[n] = \omega_X[-n]$ (remember that X is smooth) we have

$$\begin{aligned} \mathrm{Hom}(i_*\mathbb{R}_y, f_*\mathbb{R}[n]) &= \mathrm{Hom}(\mathbb{R}_y, i^! f_*\mathbb{R}_X[n]) = \mathrm{Hom}(\mathbb{R}_y, f_* i^! \mathbb{R}_X[n]) = \\ &= \mathrm{Hom}(\mathbb{R}_y, f_*\omega_F[-n]) = \mathrm{Hom}(\mathbb{R}_F, \omega_F[-n]) = H_n(F). \end{aligned}$$

The identification $H_n(F) = \mathrm{Hom}(f_*\mathbb{R}_X[n], i_*\mathbb{R}_y)$ follows similarly.

Using the identifications (20) we can rewrite the form in (19) as a pairing

$$H_n(F) \times H_n(F) \to \mathbb{R}. \tag{21}$$

The following gives the geometric significance of (19) (see [dCM02] and [JMW14, Lemma 3.4]):

Lemma 2.16. — *The form* (21) *agrees with the local intersection form* (17)*.*

From this discussion we conclude:

Proposition 2.17. — *The Decomposition Theorem holds at y if and only if the local intersection form is non-degenerate.*

2.3. The Semi-Small Index Theorem

We keep the notation of the previous section. In particular

$$f : X \to Y$$

is a semi-small map, $\dim X = n = 2m$, and $y \in Y$ is such that $F := f^{-1}(y)$ is of (half) dimension m. In the previous section we outlined a reduction of the Decomposition Theorem in the semi-small case to checking that the local intersection form on $H_n(F)$ is non-degenerate. In fact, a stronger statement is true:

Theorem 2.18 (Semi-Small Index Theorem, [dCM02]). — *The local intersection form on $H_n(F)$ is $(-1)^m$-definite.*

In this section we explain how to deduce this theorem from the Hodge-Riemann relations for semi-small classes (Theorem 2.9).

Remark 2.19. — The Semi-Small Index Theorem remains true for any proper semi-small map $f : X \to Y$, as long as X is quasi-projective and smooth [dCM05, Corollary 2.1.13]. The proof requires the Decomposition Theorem (with signs) for an arbitrary map. (One can compactify $f : X \to Y$ but one may destroy semi-smallness.)

Proof (Sketch) — Consider the composition

$$cl : H_n(F) \to H_n(X) \xrightarrow{\sim} H^n(X)$$

where the first map is induced from the inclusion $F \hookrightarrow X$ and the second map is Poincaré duality. The spaces $H_n(F)$ and $H_n(X)$ are equipped with intersection forms and $H^n(X)$ carries its Poincaré form. By basic algebraic topology:

$$cl \text{ is an isometry.} \tag{22}$$

We will use the Hodge-Riemann relations for $H^n(X)$ to deduce the index theorem. The bridge to the Hodge-Riemann relations is provided by the following two beautiful facts:

Lemma 2.20. — *Let ω denote the Chern class of $f^*\mathcal{L}$, for $\mathcal{L}$ an ample line bundle on Y. The image of cl consists of ω-primitive classes of Hodge type (m, m).*

Proof — Recall that $H_n(F)$ has a basis consisting of fundamental classes $[F_i]$ of irreducible components of $F = f^{-1}(y)$ of maximal dimension. Thus the image of cl consists of algebraic cycles, and the claim about Hodge type follows. It remains to see that the image consists of primitive classes. Under the isomorphism $H_n(X) \xrightarrow{\sim} H^n(X)$ multipliction by ω on the right corresponds to intersecting with a general hyperplane section of $f^*\mathcal{L}$ on the left. We may assume that such a hyperplane section is the inverse image, under f, of a general hyperplane section of $\mathcal{L}$. However such a hyperplane section has empty intersection with $\{y\}$ (being a point) and hence its inverse image does not intersect F. The claim follows.

Lemma 2.21. — *cl is injective.*

Proof — The pushforward $H_n(F) \to H_n(X)$ is dual to the restriction map

$$r : H^n(X) \to H^n(F).$$

We will show that r is surjective, which implies the lemma.

Let $U \subset Y$ denote an open affine neighbourhood of y. Let X_U denote the inverse image of U in X. By abuse of notation we continue to denote by f the induced map $X_U \to U$. Let $i : \{y\} \hookrightarrow U$ denote the inclusion of $\{y\}$ and j the inclusion of the complement $U \setminus \{y\}$. In the distinguished triangle

$$j_! j^! f_* \mathbb{R}_{X_U}[n] \to f_* \mathbb{R}_{X_U}[n] \to i_* i^* f_* \mathbb{R}_{X_U}[n] \xrightarrow{[1]}$$

all objects belong to ${}^pD^{\leq 0}(U)$. Because U is affine $H^q(U, \mathcal{F}) = 0$ if $q > 0$ for $\mathcal{F} \in {}^pD^{\leq 0}(U)$ by Proposition 1.8. In particular

$$r' : H^n(X_U) = H^0(U, f_*\mathbb{R}_{X_U}[n]) \to H^0(U, i_*i^*f_*\mathbb{R}_{X_U}[n]) = H^n(F)$$

is surjective.

We may factor our map r as $H^n(X) \to H^n(X_U) \xrightarrow{r'} H^n(F)$. By mixed Hodge theory [Del74, Prop. 8.2.6] the images of r and r' agree. Hence r is surjective and the lemma follows.

We may now deduce the Semi-Small Index Theorem from Theorem 2.9. We have an isometric embedding $cl : H_n(F) \hookrightarrow P^{m,m} \subset H^n(X)$. By the Hodge-Riemann relations the Poincaré form on the later space is $(-1)^m$-definite. Hence this is also the case for the intersection form on $H_n(F)$.

2.4. Hard Lefschetz via positivity

Our goal is to outline a proof of Theorem 2.9, which we will carry out in the next section. Beforehand we recall an old idea to prove the hard Lefschetz theorem by combining Poincaré duality and the weak Lefschetz theorem with the Hodge-Riemann relations in dimension one less.

To this end suppose that $X \subset \mathbb{P}$ is a smooth projective variety of dimension n and let $D \subset X$ be a general (i.e. smooth) hyperplane section. Consider the graded vector spaces

$$H = \bigoplus H^j \quad \text{where} \quad H^j := H^{n+j}(X),$$
$$H_D = \bigoplus H_D^j \quad \text{where} \quad H_D^j := H^{n-1+j}(D).$$

In the following, we attempt to carry out an inductive proof of the hard Lefschetz theorem for H. We assume as known the weak Lefschetz theorem and Poincaré duality in general and the hard Lefschetz theorem and Hodge-Riemann relations for H_D.

The inclusion $i : D \hookrightarrow X$ gives Poincaré dual restriction and Gysin morphisms

$$i^* : H^j \to H_D^{j+1} \quad \text{and} \quad i_! : H_D^j \to H^{j+1}.$$

Denote by ω the Chern class determined by our embedding $X \subset \mathbb{P}$ and let ω_D denote its restriction to D. We have:

$$\omega \wedge \alpha = i_! \circ i^*(\alpha) \quad \text{for all } \alpha \in H, \tag{23}$$

$$\omega_D \wedge \beta = i^* \circ i_!(\beta) \quad \text{for all } \beta \in H_D. \tag{24}$$

Moreover, by the weak Lefschetz theorem:

(25) $\quad i^* : H^j \to H_D^{j+1}$ is an isomorphism if $j < -1$ and injective if $j = -1$,

(26) $\quad i_! : H_D^{j-1} \to H^j$ is an isomorphism if $j > 1$ and surjective if $j = 1$.

Now the hard Lefschetz theorem for H_D implies the hard Lefschetz theorem for $\omega^k : H^{-k} \to H^k$ for $k > 1$ because we can factor ω^k as

$$H^{-k} \xrightarrow{\sim} H_D^{-k+1} \xrightarrow{\omega_D^{k-1}} H_D^{k-1} \xrightarrow{\sim} H^k$$

where the first and last maps are weak Lefschetz isomorphisms.

The missing case is $\omega : H^{-1} \to H^1$. However in this case one may use the relations (23) and (24) to deduce that i^* restricts to a map:

$$i^* : P^{-1} = \ker(\omega^2 : H^{-1} \to H^3) \to P_D^0 := \ker(\omega_D : H_D^0 \to H_D^2).$$

Hence if $0 \neq \alpha \in P^{-1}$ is of pure Hodge type (p, q) then, by weak Lefschetz and the Hodge-Riemann bilinear relations,

$$0 \neq (i^*\alpha, i^*\alpha) = (\alpha, \omega \wedge \alpha).$$

It follows that $\omega : H^{-1} \to H^1$ is injective, and hence an isomorphism (by Poincaré duality).

Remark 2.22. — The above line of reasoning can be used to deduce the Hodge-Riemann bilinear relations for all primitive subspaces $P^j \subset H^j$ with $j < 0$. However the crucial case of the Hodge-Riemann relations for the middle degree $P^0 \subset H^0$ is missing. Hence we cannot close the induction.

2.5. Hard Lefschetz and Hodge-Riemann for semi-small classes

We now outline de Cataldo and Migliorini's proof of Theorem 2.9. The basic idea is to combine the argument of the previous section with a limit argument to recover the missing Hodge-Riemann relations. Recall that

$$f : X \to Y$$

is a semi-small morphism with X connected, smooth and projective. The proof is by induction on the dimension n of X. If $n = 0, 1$ then f is finite, and the theorem can be checked by hand.

Step 1: Hard Lefschetz. Let $\mathcal{L}$ be an ample line bundle on Y, $i : D \hookrightarrow Y$ the inclusion of a general hyperplane section, $f_D : X_D := f^{-1}(D) \to D$ the induced map, $\omega \in H^2(X)$ the Chern class of $f^*\mathcal{L}$, and ω_D its restriction of X_D.

A Bertini type argument (see [dCM02, Prop. 2.1.7]) guarantees that X_D is smooth and that f_D is semi-small. Hence we can apply induction to deduce that hard Lefschetz and the Hodge-Riemann relations hold for the action of ω_D on $H^*(X_D)$. Because $f_*\mathbb{R}_X[n]$ is perverse, the weak Lefschetz theorem holds for the restriction map

$$i^* : H^{*+n}(X) = H^*(Y, f_*\mathbb{R}_X[n]) \to H^*(Y, i_*i^*f_*\mathbb{R}_X[n]) = H^{*+n}(X_D)$$

and its dual. Now the arguments of the previous section allow us to deduce that ω satisfies hard Lefschetz on $H^*(X)$.

Step 2: Hodge-Riemann. We explain how to deduce the Hodge-Riemann relations for the crucial case $H^0 = H^n(X)$. Hodge-Riemann relations in degrees < 0 follow similarly (or alternatively one can use Step 1 and Remark 2.22).

Let η denote an ample class on X. Then $\omega + \varepsilon\eta$ belongs to the ample cone for all $\varepsilon > 0$. For $\varepsilon \geq 0$ consider the subspaces:

$$P^0_\varepsilon := \ker((\omega + \varepsilon\eta) : H^0 \to H^2),$$
$$P^{p,q}_\varepsilon := (P^0_\varepsilon)_\mathbb{C} \cap H^{p,q} \quad \text{where } p + q = n.$$

We claim that, in the Grassmannian of subspaces of H^0, we have

$$\lim_{\varepsilon \to 0} P^{p,q}_\varepsilon = P^{p,q}_0. \tag{27}$$

The left hand side is clearly contained in the right hand side. The claim now follows because both sides have dimension $\dim H^{p,q} - \dim H^{p+1,q+1}$ (for the left hand side this follows via classical Hodge theory and Remark 1.6, for the right hand side it follows by hard Lefschetz for ω established in Step 1).

Recall our Hermitian form $\kappa(\alpha, \beta) = (\sqrt{-1})^n \int \alpha \wedge \overline{\beta}$ on $H^0_\mathbb{C}$. We conclude from (27) that any $\alpha \in P^{p,q}_0$ is a limit of classes in $P^{p,q}_\varepsilon$ as $\varepsilon \to 0$. Hence, by the Hodge-Riemann relations for the classes $\omega + \varepsilon\eta$ (which lie in the ample cone) we have

$$(\sqrt{-1})^{p-q-n}(-1)^{n(n-1)/2}\kappa(\alpha, \overline{\alpha}) \geq 0 \quad \text{for any } \alpha \in P^{p,q}_0. \tag{28}$$

By Hard Lefschetz the restriction of κ to each $P^{p,q}_0$ is non-degenerate. However (28) tells us that our Hermitian form is also semi-definite on $P^{p,q}_0$. We conclude that our form is definite and we have a strict inequality

$$(\sqrt{-1})^{p-q-n}(-1)^{n(n-1)/2}\kappa(\alpha, \overline{\alpha}) > 0 \quad \text{for any } \alpha \in P^{p,q}_0. \tag{29}$$

This yields the Hodge-Riemann relations for H^0.

3. GENERAL MAPS

In this section we outline de Cataldo and Migliorini's proof of the Decomposition Theorem for general projective maps. The proof follows the same main lines as the semi-small case, however the collection of statements needed through the induction is more involved. We refer to this collection as the "Decomposition Theorem Package". We begin by stating all theorems constituting the package, and then proceed to an outline of the inductive proof.

3.1. The Decomposition Theorem package

We assume as always that X is a smooth connected projective variety of dimension n and that

$$f : X \to Y$$

is a surjective projective morphism. Let us fix the following two classes in $H^2(X)$:

η : the Chern class of a relatively ample (with respect to f) line bundle,

β : the Chern class of the pull-back (via f) of an ample line bundle on Y.

Because $H^2(X) = \mathrm{Hom}_{D^b_c(X)}(\mathbb{R}_X, \mathbb{R}_X[2])$ we may interpret η as a map $\eta : \mathbb{R}_X[n] \to \mathbb{R}_X[n+2]$. Pushing forward we obtain a map (also denoted η):

$$\eta : f_*\mathbb{R}_X[n] \to f_*\mathbb{R}_X[n+2].$$

Recall that every object in $D^b_c(Y)$ carries a perverse filtration. Moreover this filtration is preserved by any morphism in $D^b_c(Y)$. Thus η induces maps (for all $m \in \mathbb{Z}$):

$$\eta : {}^p\tau_{\le m} f_*\mathbb{R}_X[n] \to {}^p\tau_{\le m+2} f_*\mathbb{R}_X[n]. \tag{30}$$

The Relative Hard Lefschetz Theorem concerns the associated graded of η:

Theorem 3.1 (Relative Hard Lefschetz Theorem). — *For $i \ge 0$, η induces an isomorphism*

$$\eta^i : {}^p\mathcal{H}^{-i}(f_*\mathbb{R}_X[n]) \xrightarrow{\sim} {}^p\mathcal{H}^{i}(f_*\mathbb{R}_X[n]).$$

Remark 3.2. — The Relative Hard Lefschetz Theorem specialises to the Hard Lefschetz Theorem if Y is a point. If f is smooth (the setting of Deligne's Theorem) the Relative Hard Lefschetz Theorem follows from the classical Hard Lefschetz Theorem applied to the fibres of f. If f is semi-small then ${}^p\mathcal{H}^{-i}(f_*\mathbb{R}_X[n]) = 0$ unless $i = 0$ and the Relative Hard Lefschetz Theorem holds trivially.

It is a formal consequence of the Relative Hard Lefschetz Theorem that we have a decomposition (see [Del91])

$$f_*\mathbb{R}_X[n] \cong \bigoplus {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])[-i]. \tag{31}$$

The heart of the Decomposition Theorem is now:

Theorem 3.3 (Semi-Simplicity Theorem). — *Each ${}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])$ is a semi-simple perverse sheaf.*

Remark 3.4. — If f is smooth (the setting of Deligne's Theorem) the Semi-Simplicity Theorem follows from the fact ([Gri70, Theorem 7.1], [Del71, Theorem 4.2.6], [Sch73, Theorem 7.25]) that a local system underlying a polarisable pure variation of Hodge structure on a smooth variety is semi-simple. If f is semi-small then all the content of the Decomposition Theorem is contained in the Semi-Simplicity Theorem for ${}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]) = f_*\mathbb{R}_X[n]$.

Remark 3.5. — By the Semi-Simplicity Theorem, we have a canonical isomorphism

$$ {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n]) = \bigoplus V_{\lambda,\mathcal{L},i} \otimes IC(\overline{Y}_\lambda, \mathcal{L}) $$

where the sum runs over all pairs $(Y_\lambda, \mathcal{L})$ consisting of a stratum Y_λ and a(n) (isomorphism class of) simple local system $\mathcal{L}$ on Y_λ, and $V_{\lambda,\mathcal{L},i}$ is a real vector space. By semi-simplicity the map

$$ \eta : {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n]) \to {}^p\mathcal{H}^{i+2}(f_*\mathbb{R}_X[n]) $$

is completely described by maps of vector spaces $\eta : V_{\lambda,\mathcal{L},i} \to V_{\lambda,\mathcal{L},i+2}$ for all pairs $(Y_\lambda, \mathcal{L})$. The Relative Hard Lefschetz theorem now becomes the statement that the degree two endomorphism η of the finite-dimensional graded vector space

$$ V_{\lambda,\mathcal{L}} := \bigoplus V_{\lambda,\mathcal{L},i} $$

satisfies hard Lefschetz for all pairs $(Y_\lambda, \mathcal{L})$.

As in the semi-small case it is important to understand the structure that the above theorems give on the global cohomology of X. We set $H^i := H^{n+i}(X) = H^i(Y, f_*\mathbb{R}_X[n])$ as usual. By taking global cohomology of the perverse filtration

$$ \dots \to {}^p\tau_{\le m} f_*\mathbb{R}_X[n] \to {}^p\tau_{\le m+1} f_*\mathbb{R}_X[n] \to \dots $$

we obtain the (global) *perverse filtration* on H:

$$ \dots \subset H_{\le m} \subset H_{\le m+1} \subset \dots $$

Recall that H is equipped with its Poincaré form. With respect to this form one has

$$ H_{\le i}^{\perp} = H_{<-i}. \tag{32} $$

Consider the asssociated graded of the perverse filtration:

$$ H_i := H_{\le i}/H_{<i} \quad \text{and} \quad \operatorname{gr} H = \bigoplus H_i. $$

By (32) the Poincaré form induces a non-degenerate form $H_i \times H_{-i} \to \mathbb{R}$ and hence a non-degenerate form on $\operatorname{gr} H$.

Proposition 3.6. — *The perverse filtration is a filtration by pure Hodge substructures. In particular, each H_i^j is a pure Hodge structure of weight $n + j$.*

The action of η and β on H satisfies:

$$ \beta(H_{\le m}) \subset H_{\le m} \quad \text{for all } m \in \mathbb{Z}, \tag{33} $$

$$ \eta(H_{\le m}) \subset H_{\le m+2} \quad \text{for all } m \in \mathbb{Z}. \tag{34} $$

(The first inclusion follows because the cohomology of any complex on Y is a graded module over $H^*(Y)$. The second inclusion follows from (30).) Hence we obtain operators

$$ \beta : H_i^j \to H_i^{j+2} \quad \text{and} \quad \eta : H_i^j \to H_{i+2}^{j+2}. $$

Theorem 3.7 (Relative Hard Lefschetz in Cohomology). — *For $i \geq 0$, η induces an isomorphism*

$$\eta^i : H_{-i} \xrightarrow{\sim} H_i. \tag{35}$$

Theorem 3.8 (Hard Lefschetz for Perverse Cohomology). — *For all $i \in \mathbb{Z}$ and $j \geq 0$, β induces an isomorphism*

$$\beta^j : H_i^{i-j} \xrightarrow{\sim} H_i^{i+j}. \tag{36}$$

Remark 3.9. — One may depict the H_i^j and maps β and η as a two-dimensional array:

$$\begin{tikzcd}
 & \ddots & & H_0^2 & & \iddots & \\
 & & H_{-1}^0 & H_0^1 & H_1^2 & & \\
 j-i \uparrow & H_{-2}^{-2} \arrow[rr, bend left, "\eta"] & H_{-1}^{-1} & H_0^0 \arrow[u, bend left, "\beta", to=1-4] \arrow[rr, bend left, "\eta"] & H_1^1 & H_2^2 & \\
 & & H_{-1}^{-2} & H_0^{-1} & H_1^0 & & \\
 & \iddots & & H_0^{-2} \arrow[uu, bend left, "\beta"] & & \ddots & \\
 & & & \xrightarrow{\quad i \quad} & & &
\end{tikzcd} \tag{37}$$

(We have only depicted the maps with source or target H_0^0.) Relative Hard Lefschetz states that η satisfies Hard Lefschetz along each row, and Hard Lefschetz for Perverse Cohomology states that β satisfies Hard Lefschetz along each column.

Recall that the hard Lefschetz theorem leads to a primitive decomposition of cohomology. The above two theorems lead to a bigraded primitive decomposition; set

$$P_{-i}^{-j} := \ker(\eta^{i+1} : H_{-i}^{-i-j} \to H_{i+2}^{i-j+2}) \cap \ker(\beta^{j+1} : H_{-i}^{-i-j} \to H_{-i}^{-i+j+2}) \subset H_{-i}^{-i-j}.$$

Corollary 3.10 ((η, β)-Primitive Decomposition). — *The inclusions $P_{-i}^{-j} \hookrightarrow H_{-i}^{-i-j}$ induce a canonical isomorphism of $\mathbb{R}[\eta, \beta]$-modules:*

$$\bigoplus_{i,j \geq 0} \mathbb{R}[\eta]/(\eta^{i+1}) \otimes \mathbb{R}[\beta]/(\beta^{j+1}) \otimes P_{-i}^{-i-j} \xrightarrow{\sim} \operatorname{gr} H.$$

Remark 3.11. — Recall that the Hard Lefschetz Theorem can be rephrased in terms of an $\mathfrak{sl}_2$-action (see Remark 1.3). Similarly, Theorems 3.7 and 3.8 are equivalent to the existence of an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-action on $\operatorname{gr} H$ such that, for all $x \in H_i^j$, we have

$$e_1(x) = \eta(x), \quad h_1(x) = i(x), \quad e_2(x) = \beta(x), \quad h_2(x) = (j-i)(x)$$

(the subscript indicates in which copy of $\mathfrak{sl}_2$ the generator lives). The (η, β)-decomposition coincides with the isotypic decomposition and the primitive subspaces P^{-j}_{-i} are the lowest weight spaces.

Our form on $\operatorname{gr} H$ induces a form on each H^{-i-j}_{-i} for $i, j \geq 0$; if $[\alpha], [\beta] \in H^{-i-j}_{-i}$ are represented by classes $\alpha, \beta \in H^{-i-j}_{\leq -i}$ we set

$$S_{ij}(\alpha, \beta) := \int_X \eta^i \wedge \beta^j \wedge \alpha \wedge \beta.$$

This form is well defined and non-degenerate.

Theorem 3.12 ((η, β)-Hodge-Riemann Bilinear Relations)
Each Hodge structure P^{-j}_{-i} (of weight $n - i - j$) is polarised by the forms S_{ij}.

Remark 3.13. — With the notation in Remark 3.5 we have

$$\operatorname{gr} H = \bigoplus V_{\lambda,\mathcal{L}} \otimes IH(\overline{Y}_\lambda, \mathcal{L}). \tag{38}$$

In the array (37) the columns (resp. rows) correspond to the grading on $IH(\overline{Y}_\lambda, \mathcal{L})$ (resp. $V_{\lambda,\mathcal{L}}$). The above theorems can be understood as saying that "each row and column looks like the cohomology of a smooth projective variety". This remarkable point of view is emphasised in [Mac83]. See [GM82] and [dCM05, §2] for examples.

Remark 3.14. — Any choice of an isomorphism of complexes as in (31) gives an isomorphism $H \cong \operatorname{gr} H$ of vector spaces. It is possible to choose this isomorphism so as to obtain an isomorphism of Hodge structures [dCM05b]. Under such an isomorphism the decomposition $H = \bigoplus V_{\lambda,\mathcal{L}} \otimes IH(\overline{Y}_\lambda, \mathcal{L})$ given by (38) is of motivic nature [dCM15]; that is, the projectors in this decomposition are motivated cycles in the sense of André [And96], and are given by algebraic cycles if Grothendieck's standard conjectures are true.

3.2. Defect of semi-smallness and structure of the proof

We now give an outline of the argument. For a projective and surjective map $f : X \to Y$ with $\dim X = n$ its *defect of semi-smallness* is

$$r(f) := \max\{i \in \mathbb{Z} \mid {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n]) \neq 0\}.$$

Equivalently, if $Y := \bigsqcup Y_\lambda$ is a stratification of f and we choose a point $y_\lambda \in Y_\lambda$ in each stratum then

$$r(f) = \max_{\lambda \in \Lambda}\{\, 2 \dim f^{-1}(y_\lambda) + \dim Y_\lambda - \dim X\}.$$

We have $r(f) \geq 0$ and $r(f) = 0$ if and only if f is semi-small.

The proof is via simultaneous induction on the defect of semi-smallness and on the dimension of the image of f. That is, if we fix f we may assume that the Decomposition Theorem Package is known for any projective map $g : X' \to Y'$ with $r(g) < r(f)$ or $r(g) = r(f)$ and $\dim g(X') < \dim f(X)$. The base case is when f is the projection to a point, in which case all statements follow from classical Hodge theory.

The proof breaks up into four main steps. The titles in italics are those of the upcoming sections. Only steps 2) and 3) have an analogue in the case of a semi-small map:

1. *Relative Hard Lefschetz via semi-simplicity.* The Relative Hard Lefschetz Theorem is deduced from the Decomposition Theorem for the relative universal hyperplane section morphism. The decomposition
$$f_*\mathbb{R}_X[n] \cong \bigoplus {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])[-i]$$
is an immediate consequence, as is the semi-simplicity of ${}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])$ for $i \neq 0$.
2. *Miraculous approximability.* All global statements in the Decomposition Theorem Package are established. The crucial case P_0^0 is established by a limit argument.
3. *Local study of the Decomposition Theorem.* Analogously to the case of a semi-small map, the (η, β)-Hodge-Riemann Bilinear Relations are used to show that we have a splitting
$$ {}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]) = \bigoplus IC(\overline{Y}_\lambda, \mathcal{L}_\lambda) \tag{39}$$
for certain local systems $\mathcal{L}_\lambda$.
4. *Semi-simplicity of local systems.* Deligne's theorem is used to show that each local system $\mathcal{L}_\lambda$ appearing in (39) is semi-simple.

Remark 3.15. — After the first step above the Semi-Simplicity Theorem is easily reduced to proving the semi-simplicity of the middle primitive summand
$$\mathcal{P}^0 := \ker(\eta : {}^p\mathcal{H}^0(f_*\mathbb{R}[n]) \to {}^p\mathcal{H}^2(f_*\mathbb{R}[n])).$$
Philosophically, the proof of the general case should involve simply repeating the proof of the semi-small case for the summand $\mathcal{P}^0$. This point of view is explained in [dCM09, §3.3.2] where de Cataldo and Migliorini call $\mathcal{P}^0$ the "semi-small soul" of the map f.

Remark 3.16. — In the approach of Beilinson-Bernstein-Deligne-Gabber and Saito the Relative Hard Lefschetz Theorem and Decomposition Theorem are deduced from purity; in their approach one gets the Decomposition Theorem "all at once". The situation is quite different in de Cataldo and Migliorini's proof, where the Relative Hard Lefschetz Theorem (deduced from the Semi-Simplicity Theorem for a map with smaller defect of semi-smallness) is a crucial stepping stone in the induction.

3.3. Relative Hard Lefschetz via semi-simplicity

We have explained in §2.4 how the Hard Lefschetz Theorem and the Hodge-Riemann Relations in dimensions $\leq n-1$ imply the hard Lefschetz theorem in dimension n. This step relies on positivity in a crucial way. In this section we explain an older approach (due to Lefschetz) which deduces hard Lefschetz from a semi-simplicity statement. This approach is used to prove the Relative Hard Lefschetz Theorem.

We first recall the idea in the absolute case. Suppose that $X \subset \mathbb{P}$ is a smooth projective variety of dimension n. We set $H^i := H^{n+i}(X)$ and let $\eta \in H^2(X)$ denote the Chern class of the embedding. We have explained in § 2.4 why weak Lefschetz and the Hard Lefschetz Theorem for a smooth hyperplane section allow us to deduce hard Lefschetz for H, except for the crucial "middle" case:

$$\eta : H^{-1} \to H^1. \tag{40}$$

If $D \subset X$ denotes a smooth hyperplane section, we also explained that we may factor (40) as the composition of the restriction and its dual:

$$\eta : H^{-1} \overset{i^*}{\hookrightarrow} H^{n-1}(D) \overset{i_!}{\twoheadrightarrow} H^1. \tag{41}$$

We now give a geometric description of the image of i^*.

Let $\mathbb{P}^\vee$ denote the complete linear system of hyperplane sections of X and let $\mathcal{Y} \subset \mathbb{P}^\vee$ denote the open subvariety of smooth hyperplane sections. The morphism

$$g : \mathcal{X} := \{(x, s) \in X \times \mathcal{Y} \mid s(x) = 0\} \to \mathcal{Y}$$

induced by the projection is the *universal hyperplane section morphism.* Its fibres are the smooth hyperplane sections of X. The map g is smooth and proper and

$$\mathcal{L} := R^{n-1} g_* \mathbb{R}_X$$

is a local system whose fibre at $D' \in \mathcal{Y}$ is $H^{n-1}(D')$.

Recall our chosen hyperplane D from above. Regarding $D \in \mathcal{Y}$ as a basepoint, we may alternately view $\mathcal{L}$ as providing us with a representation of the fundmental group $\pi_1 := \pi_1(\mathcal{Y}, D)$ on $H^{n-1}(D)$. The following two results are fundamental observations of Lefschetz. For a modern proof see [Del80, §4].

Proposition 3.17. — *We have a commutative diagram:*

$$\begin{tikzcd}
H^{-1} \arrow[rr, hook, "i^*"] \arrow[dr, "\sim"'] & & H^{n-1}(D) \arrow[rr, two heads, "i_!"] \arrow[dr] & & H^1 \\
& H^{n-1}(D)^{\pi_1} \arrow[ur, hook] & & H^{n-1}(D)_{\pi_1} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

(where V^{π_1} and V_{π_1} denote π_1-invariants and coinvariants respectively).

Corollary 3.18. — *If $H^{n-1}(D)$ is semi-simple as a π_1-module then $\eta : H^{-1} \to H^1$ is an isomorphism.*

We now return to our setting of $f : X \to Y$ a projective morphism with X smooth and projective of dimension n. For simplicity we fix an embedding $X \subset \mathbb{P}$ of X into a projective space of dimension d and let $\mathbb{P}^\vee$ denote the dual projective space. Consider the following spaces and maps:

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times \mathbb{P}^\vee \arrow[l, "p"'] \arrow[d, "f"] & \mathcal{X} := \{(x,s) \in X \times \mathbb{P}^\vee \mid s(x) = 0\} \arrow[l, "i"'] \arrow[dl, "g"] \\ Y & \mathcal{Y} := Y \times \mathbb{P}^\vee \arrow[l, "p"'] & \end{tikzcd}$$

(Note that different arrows have the same name.) The map g is the *(relative) universal hyperplane section morphism*. Its fibre over a point $(y,s) \in \mathcal{Y}$ is the intersection of $f^{-1}(y)$ with the hyperplane section of X determined by s.

The following crucial lemma ("the defect of semi-smallness goes down") allows us to apply our inductive assumptions to conclude that the Decomposition Theorem Package holds for g:

Lemma 3.19. — *If $r(f) > 0$ then $r(g) < r(f)$. If $r(f) = 0$ then $r(g) = 0$.*

The proof is an easy analysis of a stratification of g, see [dCM02, Lemma 4.7.4].

Set $m := n + d - 1 = \dim \mathcal{X}$.

Proposition 3.20 (Relative weak Lefschetz for perverse sheaves)

1. *For $j < -1$ there is a natural isomorphism:*

$$p^*({}^p\mathcal{H}^j(f_*\mathbb{R}_X[n]))[d] = {}^p\mathcal{H}^j(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]) \xrightarrow{\sim} {}^p\mathcal{H}^{j+1}(g_*\mathbb{R}_{\mathcal{X}}[m]).$$

2. *For $j > 1$ there is a natural isomorphism:*

$${}^p\mathcal{H}^{j-1}(g_*\mathbb{R}_{\mathcal{X}}[m]) \xrightarrow{\sim} {}^p\mathcal{H}^j(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]) = p^*({}^p\mathcal{H}^j(f_*\mathbb{R}_X[n]))[d].$$

3. *$p^*({}^p\mathcal{H}^{-1}(f_*\mathbb{R}_X[n]))[d]$ is the largest subobject of ${}^p\mathcal{H}^0(g_*\mathbb{R}_{\mathcal{X}}[m])$ coming from Y.*
4. *$p^*({}^p\mathcal{H}^1(f_*\mathbb{R}_X[n]))[d]$ is the largest quotient of ${}^p\mathcal{H}^0(g_*\mathbb{R}_{\mathcal{X}}[m])$ coming from Y.*

For a proof of these statements, see [BBD, 5.4.11]. The first two statements are a consequence of the fact that the restriction of f to the complement of $\mathcal{X} \subset X \times \mathbb{P}^\vee$ is affine, combined with the cohomological dimension of affine morphisms. The second two statements are relative analogues of Proposition 3.17. (The notion of largest subobject or quotient coming from Y is well defined because p is a smooth morphism with connected fibres. Thus $p^*[d]$ identifies the category of perverse sheaves on Y with a full subcategory of perverse sheaves on $\mathcal{Y}$, see [BBD, §4.2.5–6].)

Remark 3.21. — The semi-simplicity of ${}^p\mathcal{H}^j(f_*\mathbb{R}_X)$ for $j \neq -1, 0, 1$ is an immediate consequence of (1) and (2) above. The semi-simplicity of ${}^p\mathcal{H}^j(f_*\mathbb{R}_X)$ for $j = -1, 1$ follows from (3) and (4).

Let us now explain the proof of the Relative Hard Lefschetz Theorem, following [BBD, §5.4.10]. The adjunction morphism

$$\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1] \to i_*i^*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1] = i_*\mathbb{R}_{\mathcal{X}}[m+1]$$

induces morphisms

$$i^* : {}^p\mathcal{H}^j(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]) \to {}^p\mathcal{H}^{j+1}(f_*\mathbb{R}_{\mathcal{X}}[m]).$$

For $j < 0$ these are the morphisms appearing in parts (1) and (3) of Proposition 3.20. Taking duals we obtain morphisms (we use that $\mathcal{X} \subset X \times \mathbb{P}^\vee$ is smooth)

$$i_! : {}^p\mathcal{H}^{j-1}(f_*\mathbb{R}_{\mathcal{X}}[m]) \to {}^p\mathcal{H}^j(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]).$$

For $j > 0$ these are the morphisms appearing in (2) and (4) of Proposition 3.20.

For $j \geq 0$ consider the morphisms:

$${}^p\mathcal{H}^{-j-1}(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]) \xrightarrow{i^*} {}^p\mathcal{H}^{-j}(f_*\mathbb{R}_{\mathcal{X}}[m]) \xrightarrow{\eta^j} {}^p\mathcal{H}^j(f_*\mathbb{R}_{\mathcal{X}}[m]) \xrightarrow{i_!} {}^p\mathcal{H}^{j+1}(f_*\mathbb{R}_{X\times\mathbb{P}^\vee}[m+1]).$$

We claim that for $j \geq 0$ the composition is an isomorphism. If $j > 0$ this follows because the first and last maps are isomorphisms by Proposition 3.20 and the middle map is an isomorphism by the Relative Hard Lefschetz Theorem for g. For $j = 0$ the composition is an isomorphism by parts (3) and (4) of Proposition 3.20 and the Semi-Simplicity Theorem for ${}^p\mathcal{H}^0(f_*\mathbb{R}_{\mathcal{X}}[m])$ (which is known by induction).

Finally, the above composition agrees up to shift with the pull-back via p of the map

$$\eta^j : {}^p\mathcal{H}^{-j-1}(f_*\mathbb{R}_X[n]) \to {}^p\mathcal{H}^{j+1}(f_*\mathbb{R}_X[n]).$$

Hence η^j is an isomorphism for $j \geq 0$ and the Relative Hard Lefschetz Theorem follows.

Remark 3.22. — One may also use Proposition 3.20 and induction to deduce the Hard Lefschetz Theorem for Perverse Cohomology (Theorem 3.8) for all degrees except perverse degree zero. Indeed, if $\mathcal{F}$ is a complex of sheaves on Y and $\beta \in H^2(Y)$ is an ample class, then β satisfies hard Lefschetz on $H(Y,\mathcal{F})$ if and only if $\beta + \zeta \in H^2(Y \times \mathbb{P}^\vee)$ satisfies hard Lefschetz on $H(Y \times \mathbb{P}^\vee, p^*\mathcal{F}[d]) = H(Y,\mathcal{F}) \otimes H(\mathbb{P}^\vee)$, where ζ denotes the pull-back of any non-zero element of $H^2(\mathbb{P}^\vee)$.

3.4. Miraculous approximability

In this section we discuss the inductive proofs of the global statements in the Decomposition Theorem package. Here the proofs are often routine and sometimes technical and we will not attempt to give more than a rough outline. For more detail the reader is referred to [dCM05, §5.2-5.4].

What do we know at this stage? The Relative Hard Lefschetz Theorem proved in the previous step implies immediately the Relative Hard Lefschetz Theorem in Cohomology (Theorem 3.7). Also, the previous step gives the Hard Lefschetz Theorem in Perverse Cohomology (Theorem 3.8) except for perverse degree zero (i.e. $i = 0$ in Theorem 3.8) by Remark 3.22.

As in the semi-small case, an argument involving a generic hyperplane section $D \subset Y$ and the (η, β)-Hodge-Riemann relations for the restriction of f to $f^{-1}(D)$ allows us to deduce the missing $i = 0$ case of Theorem 3.8. (This technique could also be used to prove Theorem 3.8 in the other cases, and avoid Remark 3.22.) Theorem 3.7 and Theorem 3.8 and some linear algebra imply the (η, β)-Primitive Decomposition (Corollary 3.10).

All that remains are the (η, β)-Hodge-Riemann Bilinear Relations. To make sense of these relations we need Proposition 3.6, which tells us that the perverse filtration on H and its subquotients are pure Hodge structures. In de Cataldo and Migliorini's original proof this fact was deduced from Theorem 3.8. The idea is that one can use hard Lefschetz on each H_i to give a purely linear algebraic definition of the perverse filtration (as a "weight filtration" associated to the operator β), which then implies that it is linear algebraic in nature, and hence is a filtration by pure Hodge structures.

However a more recent theorem of de Cataldo and Migliorini [dCM10] gives a conceptually and practically superior proof of Proposition 3.6. Their result is that the perverse filtration of any complex on Y is given (up to shift) by a "flag filtration" associated to a general flag of closed subvarieties of Y. As a consequence the perverse filtration associated to any map is by mixed Hodge structures (this result is independent of the Decomposition Theorem and even holds over the integers). Proposition 3.6 is an easy consequence.

It remains to discuss the proof of the (η, β)-Hodge-Riemann relations (Theorem 3.12). By taking hyperplane sections in X one may deduce the (η, β)-Hodge-Riemann Bilinear Relations for the primitive subspaces $P^{-j}_{-i} \subset H^{-i-j}_{-i}$ for all $i, j \geq 0$ with $(i, j) \neq (0, 0)$. This reduction is formally analagous to the semi-small case.

In the semi-small case the missing Hodge-Riemann relations in degree 0 were deduced via a limit argument. Here the approach is similar but more involved. The complication is that P^0_0 is a subquotient of H, and so it is a priori not clear how to realise P^0_0 as a "limit" of a subspace in H. That this is still possible explains the title of this section.

We proceed as follows. For $\varepsilon > 0$, $\beta + \varepsilon\eta \in H^2(X)$ lies in the ample cone. Hence if

$$\Lambda_\varepsilon := \ker(\beta + \varepsilon\eta : H^0 \to H^2) \subset H^0 = H^n(X)$$

then $d := \dim \Lambda_\varepsilon = \dim H^0 - \dim H^2$ by hard Lefschetz for $\beta + \varepsilon\eta$ (see Remark 1.6). Consider the limit (taken in the Grassmannian of d-dimensional subspaces of H^0):

$$\Lambda := \lim_{\varepsilon \to 0} \Lambda_\varepsilon.$$

Each Λ_ε is a polarised Hodge substructure of H^0. Hence Λ is a Hodge substructure (being a Hodge substructure is a closed condition). Also, Λ is semi-polarised (i.e. the Hodge-Riemann relations hold for Λ if we replace strict inequality > 0 by ≥ 0).

To keep track of degrees let us denote the map $\beta : H^{-i} \to H^{-i+2}$ by β_i. Of course $\Lambda \subset \ker(\beta_0)$ however equality does not hold in general because

$$\dim \ker(\beta_0) = \dim \Lambda + \dim \ker(b_2) \tag{42}$$

as follows, for example, by noticing that one can perform this calculation on $\mathrm{gr}\, H$.

It is important to be able to identify $\Lambda \subset H^0$ intrinsically. This is completed in [dCM05, §5.4]. As a consequence they deduce:

Lemma 3.23. — *We have an orthogonal decomposition* $\ker \beta_0 = \Lambda \oplus \eta(\ker \beta_2)$.

The Poincaré form on the image of $\ker \beta_0$ in H^0_0 is non-degenerate by the Hard Lefschetz Theorem for H_0. Thus the radical of the Poincaré form on $\ker \beta_0$ is $\ker \beta_0 \cap H^0_{<0}$. It follows from Lemma 3.23 that the radical of the Poincaré form on Λ is

$$\Lambda_{<0} := \Lambda \cap H^0_{<0}.$$

Thus the Poincaré form on

$$\Lambda_0 := \Lambda / \Lambda_{<0}$$

is a non-degenerate semi-polarisation, and thus a polarisation.

Finally, with a little more work the above lemma also implies that we have an embedding of Hodge structures

$$P_0^0 \hookrightarrow \Lambda_0$$

which proves the missing Hodge-Riemann relations for P_0^0. (A summand of a polarised Hodge structure is polarised.)

3.5. Local study of the Decomposition Theorem

Our induction so far gives a decomposition

$$f_*\mathbb{R}_X[n] \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])[-i] \tag{43}$$

and we know that each ${}^p\mathcal{H}^i(f_*\mathbb{R}_X[n])$ is semi-simple for $i \neq 0$. It remains to deduce the semi-simplicity of ${}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])$. In this section we outline the proof of the following:

Proposition 3.24. — *There exists a local system $\mathcal{L}_\lambda$ on each stratum Y_λ such that we have a canonical isomorphism*

$$ {}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]) = \bigoplus_{\lambda \in \Lambda} IC(\overline{Y_\lambda}, \mathcal{L}_\lambda). \tag{44}$$

In the next section we explain why each $\mathcal{L}_\lambda$ is semi-simple, which completes the proof of the Semi-Simplicity Theorem, and hence of the Decomposition Theorem.

As in the semi-small case one can reduce (by taking normal slices) to the case of a point stratum $\{y\}$. Denote by $i : \{y\} \hookrightarrow Y$ the inclusion. Let us say that ${}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])$ is *semi-simple at y* if

$$ {}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]) = i_*V \oplus \mathcal{F} \tag{45}$$

where i_*V is the skyscraper sheaf at y with stalk $V := H^0({}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])_y)$. (As in the semi-small case, $\mathcal{F}$ is a perverse sheaf whose structure can be ignored.)

Remark 3.25. — As in the semi-small case (45) is the key to establishing (44). By induction we may assume that the restriction of $\mathcal{F}$ to the complement of all point strata is a direct sum of intersection cohomology complexes as in (44), and (45) allows us to extend this decomposition over point strata.

Exactly as earlier we consider the form

$$\mathrm{Hom}(i_*\mathbb{R}_y, f_*\mathbb{R}_X[n]) \times \mathrm{Hom}(f_*\mathbb{R}_X[n], i_*\mathbb{R}_y) \to \mathrm{End}(i_*\mathbb{R}_y) = \mathbb{R}$$

and again this form is canonically identified with the intersection form

$$Q : H_n(f^{-1}(y)) \times H_n(f^{-1}(y)) \to \mathbb{R} \tag{46}$$

given by the embedding $f^{-1}(y) \hookrightarrow X$.

Remark 3.26. — It will become clear in the discussion below that the rank of this form is precisely the multiplicity of $i_*\mathbb{R}_y$ in ${}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])$. However now the situation is considerably more complicated because it is difficult to predict a priori what this rank should be. (This should be contrasted with the semi-small case where we knew that this multiplicity is always $\dim H_n(f^{-1}(y))$.) As far as the Decomposition Theorem is concerned, this is the fundamental difference between a semi-small and a general map.

The technology of perverse sheaves provides a formal means of circumventing this obstacle. Via the identifications (see Lemma 2.15)

$$H_n(f^{-1}(y)) = \mathrm{Hom}(i_*\mathbb{R}_y, f_*\mathbb{R}_X[n]) = H^0(i^! f_*\mathbb{R}_X[n]) \tag{47}$$

the perverse filtration induces a filtration on $i^! f_*\mathbb{R}_X[n]$, and hence on $H_n(f^{-1}(y))$:

$$H_{n,\le i}(f^{-1}(y)) := \mathrm{im}(H^0(i^!({}^p\tau_{\le i} f_*\mathbb{R}_X[n])) \to H^0(i^! f_*\mathbb{R}_X[n])).$$

We continue to refer to this as the *perverse filtration*:

$$\cdots \subset H_{n,\le -1}(f^{-1}(y)) \subset H_{n,\le 0}(f^{-1}(y)) \subset \ldots$$

It turns out that the perverse filtration tells us precisely what the radical of the intersection form should be. From the definition of the perverse t-structure we have $H^0(i^!({}^p\tau_{>0} f_*\mathbb{R}_X[n])) = 0$. Hence:

$$H_{n,\le 0}(f^{-1}(y)) = H_n(f^{-1}(y)). \tag{48}$$

Also, as ${}^p\tau_{<0} f_*\mathbb{R}_X[n]$ does not contain any summand isomorphic to $i_*\mathbb{R}_y$ we deduce:

$$H_{n,<0}(f^{-1}(y)) \subset \mathrm{rad}\, Q \subset H_n(f^{-1}(y)). \tag{49}$$

Finally, ${}^p\mathcal{H}^0(f^*\mathbb{R}_X[n])$ is semi-simple at y if and only the composition

$$\mathrm{Hom}(i_*\mathbb{R}_y, {}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])) \times \mathrm{Hom}({}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]), i_*\mathbb{R}_y) \to \mathrm{End}(i_*\mathbb{R}_y) = \mathbb{R}$$

is non-degenerate. We conclude:

Proposition 3.27. — *${}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])$ is semi-simple at y if and only if the intersection form induces a non-degenerate form on*

$$H_{n,0}(f^{-1}(y)) := H_{n,\leq 0}(f^{-1}(y))/H_{n,<0}(f^{-1}(y)).$$

We use the (η,β)-Hodge-Riemann relations to conclude the proof:

Theorem 3.28 (Index Theorem for maps). — *The inclusion $f^{-1}(y) \hookrightarrow X$ yields an injection of pure Hodge structures*

$$H_{n,0}(f^{-1}(y)) \hookrightarrow H_0^0.$$

This is an isometry with respect to the intersection form on the left and the Poincaré form on the right. In particular, the intersection form on $H_{n,0}(f^{-1}(y))$ underlies a polarization of pure Hodge structure, and hence is non-degenerate.

After taking the perverse filtrations into account, the proof (first replace Y by an affine neighbourhood of y, and then apply mixed Hodge theory) is identical to the proof in the semi-small case.

3.6. Semi-simplicity of local systems

It remains to see that all local systems occurring in the decomposition

$$ {}^p\mathcal{H}^0(f_*\mathbb{R}_X[n]) = \bigoplus_{\lambda\in\Lambda} IC(\overline{Y}_\lambda, \mathcal{L}_\lambda) \tag{50}$$

are semi-simple. The idea is to exhibit each $\mathcal{L}_\lambda$ (or more precisely its restriction to a non-empty Zariski open subvariety $U \subset Y_\lambda$) as the quotient of a local system associated with a smooth proper map. Such local systems are semi-simple by Deligne's theorem, and the semi-simplicity of each $\mathcal{L}_\lambda$ follows. (Note that $\pi_1(U) \twoheadrightarrow \pi_1(Y_\lambda)$ is surjective, so it is enough to know that the restriction of $\mathcal{L}_\lambda$ to U is semi-simple.)

Let $Y_\lambda \subset Y$ denote a stratum of dimension s. The local system $\mathcal{L}_\lambda$ occurring on the right of (50) is

$$\mathcal{L}_\lambda := \mathcal{H}^{-s}({}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])_{Y_\lambda}).$$

We must show that $\mathcal{L}_\lambda$ is semi-simple.

By proper base change

$$H^j((f_*\mathbb{R}_X[n])_y) = H^{n-j}(f^{-1}(y)) \quad \text{for all } y \in Y.$$

The perverse filtration on $f_*\mathbb{R}_X[n]$ induces a filtration on $H(f^{-1}(y))$ which we denote by $H_{\leq m}(f^{-1}(y))$. We set

$$H_m(f^{-1}(y)) = H_{\leq m}(f^{-1}(y))/H_{<m}(f^{-1}(y)).$$

For any $y \in Y_\lambda$ we have

$$(\mathcal{L}_\lambda)_y = H^{-s}({}^p\mathcal{H}^0(f_*\mathbb{R}_X[n])_y) = H_0^{n-s}(f^{-1}(y)).$$

This equality exhibits the fibres of $\mathcal{L}_\lambda$ as subquotients of the cohomology of a variety. However we cannot apply Deligne's theorem because the fibres $f^{-1}(y)$ are typically not smooth.

A key observation is that if $D \subset Y$ denotes the intersection of s general hyperplanes through y which are transverse to all strata then $f^{-1}(D)$ is smooth and we have a surjection

$$H^{n-s}(f^{-1}(D)) \twoheadrightarrow H_0^{n-s}(f^{-1}(y)). \tag{51}$$

This observation is not difficult; the proof is similar to that of Lemma 2.21.

With a little work (see [dCM05, §6.4]) one can find a smooth family

$$g : \mathcal{X}_U \to U$$

over a Zariski open subset $U \subset Y_\lambda$ whose fibre over $y \in U$ is $f^{-1}(D)$, for some D as in the previous paragraph. The local system $R^{n-s}g_*\mathbb{R}_{\mathcal{X}_U}$ is semi-simple by Deligne's theorem and one has a surjection $R^{n-s}g_*\mathbb{R}_{\mathcal{X}_U} \twoheadrightarrow (\mathcal{L}_\lambda)_{|U}$, which on stalks gives maps as in (51). The semi-simplicity of $\mathcal{L}_\lambda$ follows.

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