Séminaire BOURBAKI Janvier 2019
71e année, 2018–2019, nO 1157

# BOUNDEDNESS RESULTS FOR SINGULAR FANO VARIETIES, AND APPLICATIONS TO CREMONA GROUPS
## [following Birkar and Prokhorov–Shramov]

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### Contents



## 1. MAIN RESULTS

Throughout this paper, we work over the field of complex numbers.

### 1.1. Boundedness of singular Fano varieties

A normal, projective variety $X$ is called *Fano* if a negative multiple of its canonical divisor class is Cartier and if the associated line bundle is ample. Fano varieties appear throughout geometry and have been studied intensely, in many contexts. For the purposes of this talk, we remark that Fanos with sufficiently mild singularities constitute one of the fundamental variety classes in birational geometry. In fact, given any projective manifold $X$, the Minimal Model Programme (MMP) predicts the existence of a sequence of rather special birational transformations, known as "divisorial contractions" and "flips", as follows,

$$X = X^{(0)} \underset{\text{birational}}{\overset{\alpha^{(1)}}{\dashrightarrow}} X^{(1)} \underset{\text{birational}}{\overset{\alpha^{(2)}}{\dashrightarrow}} \cdots \underset{\text{birational}}{\overset{\alpha^{(n)}}{\dashrightarrow}} X^{(n)}.$$

Stefan Kebekus gratefully acknowledges support through a fellowship of the Freiburg Institute of Advanced Studies (FRIAS).

The resulting variety $X^{(n)}$ is either canonically polarised (which is to say that a suitable power of its canonical sheaf is ample), or it has the structure of a fibre space whose general fibres are either Fano or have numerically trivial canonical class. The study of (families of) Fano varieties is thus one of the most fundamental problems in birational geometry.

*Remark 1.1* (Singularities). — Even though the starting variety $X$ is a manifold by assumption, it is well understood that we cannot expect the varieties $X^{(\bullet)}$ to be smooth. Instead, they exhibit mild singularities, known as "terminal" or "canonical" — we refer the reader to [KM98, Sect. 2.3] or [Kol13, Sect. 2] for a discussion and for references. If $X^{(n)}$ admits the structure of a fibre space, its general fibres will also have terminal or canonical singularities. Even if one is primarily interested in the geometry of *manifolds*, it is therefore necessary to include families of *singular* Fanos in the discussion.

In a series of two fundamental papers, [Bir16a, Bir16b], Birkar confirmed a long-standing conjecture of Alexeev and Borisov–Borisov, [Ale94, BB92], asserting that for every $d \in \mathbb{N}$, the family of $d$-dimensional Fano varieties with terminal singularities is bounded: there exists a proper morphism of quasi-projective schemes over the complex numbers, $u : \mathbb{X} \to Y$, and for every $d$-dimensional Fano $X$ with terminal singularities a closed point $y \in Y$ such that $X$ is isomorphic to the fibre $\mathbb{X}_y$. In fact, a much more general statement holds true.

Theorem 1.2 (Boundedness of $\varepsilon$-lc Fanos, [Bir16b, Thm. 1.1])
*Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, let $\mathcal{X}_{d,\varepsilon}$ be the family of projective varieties $X$ with dimension $\dim_{\mathbb{C}} X = d$ that admit an $\mathbb{R}$-divisor $B \in \mathbb{R}\,\mathrm{Div}(X)$ such that the following holds true.*

(1.2.1) *The tuple $(X, B)$ forms a pair. In other words: $X$ is normal, the coefficients of $B$ are contained in the interval $[0, 1]$ and $K_X + B$ is $\mathbb{R}$-Cartier.*
(1.2.2) *The pair $(X, B)$ is $\varepsilon$-lc. In other words, the total log discrepancy of $(X, B)$ is greater than or equal to $\varepsilon$.*
(1.2.3) *The $\mathbb{R}$-Cartier divisor $-(K_X + B)$ is nef and big.*

*Then, the family $\mathcal{X}_{d,\varepsilon}$ is bounded.*

*Remark 1.3* (Terminal singularities). — If $X$ has terminal singularities, then $(X, 0)$ is 1-lc. We refer to Section 2.3, to Birkar's original papers, or to [HMX14, Sect. 3.1] for the relevant definitions concerning more general classes of singularities.

For his proof of the boundedness of Fano varieties and for his contributions to the Minimal Model Programme, Caucher Birkar was awarded with the Fields Medal at the ICM 2018 in Rio de Janeiro.

**1.1.1.** *Where does boundedness come from?* — The brief answer is: "From boundedness of volumes!" In fact, if $(X_t, A_t)_{t\in T}$ is a family of tuples where the $X_t$ are normal, projective varieties of fixed dimension $d$ and $A_t \in \operatorname{Div}(X_t)$ are very ample, and if there exists a number $v \in \mathbb{N}$ such that

$$\operatorname{vol}(A_t) := \limsup_{n\to\infty} \frac{d! \cdot h^0\big(X_t,\, \mathscr{O}_{X_t}(n \cdot A_t)\big)}{n^d} < v$$

for all $t \in T$, then elementary arguments using Hilbert schemes show that the family $(X_t, A_t)_{t\in T}$ is bounded.

For the application that we have in mind, the varieties $X_t$ are the Fano varieties whose boundedness we would like to show and the divisors $A_t$ will be chosen as fixed multiples of their anticanonical classes. To obtain boundedness results in this setting, Birkar needs to show that there exists one number $m$ that makes all $A_t := -m \cdot K_{X_t}$ very ample, or (more modestly) ensures that the linear systems $|-m \cdot K_{X_t}|$ define birational maps. Volume bounds for these divisors need to be established, and the singularities of the linear systems need to be controlled.

**1.1.2.** *Earlier results, related results.* — Boundedness results have a long history, which we cannot cover with any pretence of completeness. Boundedness of smooth Fano surfaces and threefolds follows from their classification. Boundedness of Fano *manifolds* of arbitrary dimension was shown in the early 1990s, in an influential paper of Kollár, Miyaoka and Mori, [KMM92], by studying their geometry as rationally connected manifolds. Around the same time, Borisov–Borisov were able to handle the toric case using combinatorial methods, [BB92]. For (singular) surfaces, Theorem 1.2 is due to Alexeev, [Ale94].

Among the newer results, we will only mention the work of Hacon–M^cKernan–Xu. Using methods that are similar to those discussed here, but without the results on "boundedness of complements" (→ Section 4), they are able to bound the volumes of klt pairs $(X, \Delta)$, where $X$ is projective of fixed dimension, $K_X + \Delta$ is numerically trivial and the coefficients of $\Delta$ come from a fixed DCC set, [HMX14, Thm. B]. Boundedness of Fanos with klt singularities and fixed Cartier index follows, [HMX14, Cor. 1.8]. In a subsequent paper [HX15] these results are extended to give the boundedness result that we quote in Theorem 4.6, and that Birkar builds on. We conclude with a reference to [Jia17, Che18] for current results involving $K$-stability and $\alpha$-invariants. The surveys [HM10, HMX18] give a more complete overview.

### 1.2. Applications

As we will see in Section 8 below, boundedness of Fanos can be used to prove the existence of fixed points for actions of finite groups on Fanos, or more generally rationally connected varieties. Recall that a variety $X$ is *rationally connected* if every two points are connected by an irreducible, rational curve contained in $X$. This allows us to apply Theorem 1.2 in the study of finite subgroups of birational automorphism groups.

**1.2.1.** *The Jordan property of Cremona groups.* — Even before Theorem 1.2 was known, it had been realised by Prokhorov and Shramov, [PS16], that boundedness of Fano varieties with terminal singularities would imply that the birational automorphism groups of projective spaces (= Cremona groups, $\mathrm{Bir}(\mathbb{P}^d)$) satisfy the *Jordan property.* Recall that a group $\Gamma$ is said to *have the Jordan property* if there exists a number $j \in \mathbb{N}$ such that every finite subgroup $G \subset \Gamma$ contains a normal, Abelian subgroup $A \subset G$ of index $|G : A| \leq j$. In fact, a stronger result holds.

THEOREM 1.4 (Jordan property of Cremona groups, [Bir16b, Cor. 1.3], [PS16, Thm. 1.8]) *Given any number $d \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that for every complex, projective, rationally connected variety $X$ of dimension $\dim_{\mathbb{C}} X = d$, every finite subgroup $G \subset \mathrm{Bir}(X)$ contains a normal, Abelian subgroup $A \subseteq G$ of index $|G : A| \leq j$.*

*Remark 1.5.* — Theorem 1.4 answers a question of Serre [Ser09, 6.1] in the positive. A more detailed analysis establishes the Jordan property more generally for all varieties of vanishing irregularity, [PS14, Thm. 1.8].

Theorem 1.4 ties in with the general philosophy that finite subgroups of $\mathrm{Bir}(\mathbb{P}^d)$ should in many ways be similar to finite linear groups, where the property has been established by Jordan more then a century ago.

THEOREM 1.6 (Jordan property of linear groups, [Jor77]). — *Given any number $d \in \mathbb{N}$, there exists $j_d^{\mathrm{Jordan}} \in \mathbb{N}$ such that every finite subgroup $G \subset \mathrm{GL}_d(\mathbb{C})$ contains a normal, Abelian subgroup $A \subseteq G$ of index $|G : A| \leq j_d^{\mathrm{Jordan}}$.* □

*Remark 1.7* (Related results). — For further information on Cremona groups and their subgroups, we refer the reader to the surveys [Pop14, Can18] and to the recent research paper [Pop18b]. For the maximally connected components of automorphism groups of projective varieties (rather than the full group of birational automorphisms), the Jordan property has recently been established by Meng and Zhang without any assumption on the nature of the varieties, [MZ18, Thm. 1.4]; their proof uses group-theoretic methods rather than birational geometry. For related results (also in positive characteristic), see [Hu18, Pop18a, SV18] and references there.

**1.2.2.** *Boundedness of finite subgroups in birational transformation groups.* — Following similar lines of thought, Prokhorov and Shramov also deduce boundedness of finite subgroups in birational transformation groups, for arbitrary varieties defined over a finite field extension of $\mathbb{Q}$.

THEOREM 1.8 (Bounds for finite groups of birational transformation, [PS14, Thm. 1.4]) *Let $k$ be a finitely generated field over $\mathbb{Q}$. Let $X$ be a variety over $k$, and let $\mathrm{Bir}(X)$ denote the group of birational automorphisms of $X$ over $\mathrm{Spec}\, k$. Then, there exists $b \in \mathbb{N}$ such that any finite subgroup $G \subset \mathrm{Bir}(X)$ has order $|G| \leq b$.*

As an immediate corollary, they answer another question of Serre[(1)], pertaining to finite subgroups in the automorphism group of a field.

Corollary 1.9 (Boundedness for finite groups of field automorphisms, [PS14, Cor. 1.5]) *Let $k$ be a finitely generated field over $\mathbb{Q}$. Then, there exists $b \in \mathbb{N}$ such that any finite subgroup $G \subset \mathrm{Aut}(k)$ has order $|G| \leq b$.*

**1.2.3.** *Boundedness of links.* — Birkar's result has further applications within birational geometry. Combined with work of Choi–Shokurov, it implies the boundedness of Sarkisov links in any given dimension, cf. [CS11, Cor. 7.1].

### 1.3. Outline of this paper

Paraphrasing [Bir16a, p. 6], the main tools used in Birkar's work include the Minimal Model Programme [KM98, BCHM10], the theory of complements [PS01, PS09, Sho00], the technique of creating families of non-klt centres using volumes [HMX14, HMX13] and [Kol97, Sect. 6], and the theory of generalised polarised pairs [BZ16]. In fact, given the scope and difficulty of Birkar's work, and given the large number of technical concepts involved, it does not seem realistic to give more than a panoramic presentation of Birkar's proof here. Largely ignoring all technicalities, Sections 4–7 highlight four core results, each of independent interest. We explain the statements in brief, sketch some ideas of proof and indicate how the results might fit together to give the desired boundedness result. Finally, Section 8 discusses the application to the Jordan property in some detail.

### 1.4. Acknowledgements

The author would like to thank Florin Ambro, Serge Cantat, Enrica Floris, Christopher Hacon, Vladimir Lazić, Benjamin McDonnell, Vladimir Popov, Thomas Preu, Yuri Prokhorov, Vyacheslav Shokurov, Chenyang Xu and one anonymous reader, who answered my questions and/or suggested improvements. Yanning Xu was kind enough to visit Freiburg and patiently explain large parts of the material to me. He helped me out more than just once. His paper [Xu18], which summarises Birkar's results, has been helpful in preparing these notes. Even though our point of view is perhaps a little different, it goes without saying that this paper has substantial overlap with Birkar's own survey [Bir18].

[(1)] Unpublished problem list from the workshop "Subgroups of Cremona groups: classification", 29–30 March 2010, ICMS, Edinburgh. Available at `http://www.mi.ras.ru/~prokhoro/preprints/edi.pdf`. Serre's question is found on page 7.

## 2. NOTATION, STANDARD FACTS AND KNOWN RESULTS

### 2.1. Varieties, divisors and pairs

We follow standard conventions concerning varieties, divisors and pairs. In particular, the following notation will be used.

Definition 2.1 (Round-up, round-down and fractional part)
*If $X$ is a normal, quasi-projective variety and $B \in \mathbb{R}\operatorname{Div}(X)$ an $\mathbb{R}$-divisor on $X$, we write $\lfloor B \rfloor$, $\lceil B \rceil$ for the round-down and round-up of $B$, respectively. The divisor $\{B\} := B - \lfloor B \rfloor$ is called* fractional part *of $B$.*

Definition 2.2 (Pair). — *A* pair *is a tuple $(X, B)$ consisting of a normal, quasi-projective variety $X$ and an effective $\mathbb{R}$-divisor $B$ such that $K_X + B$ is $\mathbb{R}$-Cartier.*

Definition 2.3 (Couple). — *A* couple *is a tuple $(X, B)$ consisting of a normal, projective variety $X$ and a divisor $B \in \operatorname{Div}(X)$ whose coefficients are all equal to one. The couple is called* log-smooth *if $X$ is smooth and if $B$ has simple normal crossings support.*

### 2.2. $\mathbb{R}$-divisors

While divisors with real coefficients had sporadically appeared in birational geometry for a long time, the importance of allowing real (rather than rational) coefficients was highlighted in the seminal paper [BCHM10], where continuity- and compactness arguments for spaces of divisors were used in an essential manner. Almost all standard definitions for divisors have analogues for $\mathbb{R}$-divisors, but the generalised definitions are perhaps not always obvious. For the reader's convenience, we recall a few of the more important notions here.

Definition 2.4 (Big $\mathbb{R}$-divisors). — *Let $X$ be a normal, projective variety. A divisor $B \in \mathbb{R}\operatorname{Div}(X)$, which need not be $\mathbb{R}$-Cartier, is called* big *if there exists an an ample $H \in \mathbb{R}\operatorname{Div}(X)$, and effective $D \in \mathbb{R}\operatorname{Div}(X)$ and an $\mathbb{R}$-linear equivalence $B \sim_{\mathbb{R}} H + D$.*

Definition 2.5 (Volume of an $\mathbb{R}$-divisor). — *Let $X$ be a normal, projective variety of dimension $d$. The* volume *of an $\mathbb{R}$-divisor $D \in \mathbb{R}\operatorname{Div}(X)$ is defined as*

$$\operatorname{vol}(D) := \limsup_{m\to\infty} \frac{d! \cdot h^0\big(X,\, \mathscr{O}_X(\lfloor mD \rfloor)\big)}{m^d}.$$

Definition 2.6 (Linear system). — *Let $X$ be a normal, quasi-projective variety and let $M \in \mathbb{R}\operatorname{Div}(X)$. The* $\mathbb{R}$-linear system $|M|$ *is defined as*

$$|M|_{\mathbb{R}} := \{D \in \mathbb{R}\operatorname{Div}(X) \,|\, D \text{ is effective and } D \sim_{\mathbb{R}} M\}.$$

### 2.3. Invariants of varieties and pairs

We briefly recall a number of standard definitions concerning singularities. In brief, if $X$ is smooth, and if $\pi : \widetilde{X} \to X$ is any birational morphism, where $\widetilde{X}$ it smooth, then any top-form $\sigma \in H^0\big(X, \omega_X\big)$ pulls back to a holomorphic differential form $\tau \in H^0\big(\widetilde{X}, \omega_{\widetilde{X}}\big)$, with zeros along the positive-dimensional fibres of $\pi$. However, if $X$ is singular, if $\pi : \widetilde{X} \to X$ is a resolution of singularities and if $\sigma \in H^0\big(X, \omega_X\big)$ is any section in the (pre-)dualising sheaf, then the pull-back of $\sigma$ will only be a rational differential form on $\widetilde{X}$ which might well have poles along the positive-dimensional fibres of $\pi$. The idea in the definition of "log discrepancy" is to use this pole order to measure the "badness" of the singularities on $X$. We refer the reader to one of the standard references [KM98, Sect. 2.3] and [Kol13, Sect. 2] for an-depth discussion of these ideas and of the singularities of the Minimal Model Programme. Since the notation is not uniform across the literature[(2)], we spend a few lines to fix notation and briefly recall the central definitions of the field.

Definition 2.7 (Log discrepancy). — *Let $(X, B)$ a pair and let $\pi : \widetilde{X} \to X$ be a log resolution of singularities, with exceptional divisors $(E_i)_{1 \le i \le n}$. Since $K_X + B$ is $\mathbb{R}$-Cartier by assumptions, there exists a well-defined notion of pull-back, and a unique divisor $B_{\widetilde{X}} \in \mathbb{R}\operatorname{Div}(\widetilde{X})$ such that $K_{\widetilde{X}} + B_{\widetilde{X}} = \pi^*(K_X + B)$ in $\mathbb{R}\operatorname{Div}(\widetilde{X})$. If $D$ is any prime divisor on $\widetilde{X}$, we consider the* log discrepancy

$$a_{\log}(D, X, B) := 1 - \operatorname{mult}_D B_{\widetilde{X}}.$$

*The infimum over all such numbers,*

$$a_{\log}(X, B) := \inf\{a_{\log}(D, X, B) \mid \pi : \widetilde{X} \to X \text{ a log resolution and } D \in \operatorname{Div}(\widetilde{X}) \text{ prime}\}$$

*is called the* total log discrepancy of the pair $(X, B)$.

The total log discrepancy measures how bad the singularities are: the smaller $a_{\log}(X, B)$ is, the worse the singularities are. Table 1 on the following page lists the classes of singularities will be relevant in the sequel. In addition, $(X, B)$ is called *plt* if $a_{\log}(D, X, B) > 0$ for every resolution $\pi : \widetilde{X} \to X$ and every *exceptional* divisor $D$ on $\widetilde{X}$. The class of $\varepsilon$-lc singularities, which is perhaps the most relevant for our purposes, was introduced by Alexeev.

**2.3.1.** *Places and centres.* — The divisors $D$ that appear in the definition log discrepancy deserve special attention, in particular if $a_{\log}(D, X, B) \le 0$.

Definition 2.8 (Non-klt places and centres). — *Let $(X, B)$ a pair. A* non-klt place *of $(X, B)$ is a prime divisor $D$ on birational models of $X$ such that $a_{\log}(D, X, B) \le 0$. A* non-klt centre *is the image on $X$ of a non-klt place. When $(X, B)$ is lc, a non-klt centre is also called an* lc centre.

(2) The papers [Bir16a, Bir16b, BCHM10] denote the log discrepancy by $a(D, X, B)$, while the standard reference books [KM98, Kol13] write $a(D, X, B)$ for the standard (= "non-log") discrepancies.

| If ..., then | | $(X, B)$ is called "..." |
|---|---|---|
| $a_{\log}(X, B) \geq 0$ | ... | *log canonical (or "lc")* |
| $a_{\log}(X, B) > 0$ | ... | *Kawamata log terminal (or "klt")* |
| $a_{\log}(X, B) \geq \varepsilon$ | ... | *$\varepsilon$-log canonical (or "$\varepsilon$-lc")* |
| $a_{\log}(X, B) \geq 1$ | ... | *canonical* |
| $a_{\log}(X, B) > 1$ | ... | *terminal* |

Table 1. Singularities of the Minimal Model Programme

**2.3.2.** *Thresholds.* — Suppose that $(X, B)$ is a klt pair, and that $D$ is an effective divisor on $X$. The pair $(X, B + t \cdot D)$ will then be log-canonical for sufficiently small numbers $t$, but cannot be klt when $t$ is large. The critical value of $t$ is called the *log-canonical threshold.*

Definition 2.9 (LC threshold, compare [Laz04, Sect. 9.3.B])

*Let $(X, B)$ be a klt pair. If $D \in \mathbb{R}\,\mathrm{Div}(X)$ is effective, one defines the* lc threshold of $D$ with respect to $(X, B)$ *as*

$$\mathrm{lct}\big(X,\ B,\ D\big) := \sup\big\{t \in \mathbb{R} \bigm| (X, B + t \cdot D) \text{ is lc}\big\}.$$

*If $\Delta \in \mathbb{R}\,\mathrm{Div}(X)$ is $\mathbb{R}$-Cartier with non-empty $\mathbb{R}$-linear system (but not necessarily effective itself), one defines* lc threshold of $|\Delta|_{\mathbb{R}}$ with respect to $(X, B)$ *as*

$$\mathrm{lct}\big(X,\ B,\ |\Delta|_{\mathbb{R}}\big) := \inf\big\{\mathrm{lct}(X, B, D) \bigm| D \in |\Delta|_{\mathbb{R}}\big\}.$$

*Remark 2.10.* — In the setting of Definition 2.9, it is a standard fact that

$$\mathrm{lct}\big(X,\ B,\ |\Delta|_{\mathbb{R}}\big) = \sup\big\{t \in \mathbb{R} \bigm| (X, B + t \cdot D) \text{ is lc for every } D \in |\Delta|_{\mathbb{R}}\big\}.$$

In particular, if $(X, B)$ is klt, then $(X, B + t' \cdot D)$ is lc for every $D \in |\Delta|_{\mathbb{R}}$ and every $0 < t' < t$.

*Notation 2.11.* — If $B = 0$, we omit it from the notation and write $\mathrm{lct}\big(X,\ |\Delta|_{\mathbb{R}}\big)$ and $\mathrm{lct}\big(X,\ D\big)$ in short.

### 2.4. Fano varieties and pairs

Fano varieties come in many variants. For the purposes of this overview, the following classes of varieties will be the most relevant.

Definition 2.12 (Fano and weak log Fano pairs, [Bir16a, Sect. 2.10])

- *A projective pair $(X, B)$ is called* log Fano *if $(X, B)$ is lc and if $-(K_X + B)$ is ample. If $B = 0$, we just say that $X$ is Fano.*
- *A projective pair $(X, B)$ is called is called* weak log Fano *if $(X, B)$ is lc and $-(K_X + B)$ is nef and big. If $B = 0$, we just say that $X$ is weak Fano.*

*Remark 2.13* (Relative notions). — There exist relative versions of the notions discussed above. If $(X, B)$ is any quasi-projective pair, if $Z$ is normal and if $X \to Z$ is surjective, projective and with connected fibres, we say $(X, B)$ is log Fano over $Z$ if it is lc and if $-(K_X + B)$ is relatively ample over $Z$. Ditto with "weak log Fano".

### 2.5. Varieties of Fano type

Varieties $X$ that *admit* a boundary $B$ that makes $(X, B)$ a Fano pair are said to be of *Fano type*. This notion was introduced by Prokhorov and Shokurov in [PS09]. We refer to that paper for basic properties of varieties of Fano type.

Definition 2.14 (Varieties of Fano type, [PS09, Lem. and Def. 2.6])
*A normal, projective variety $X$ is said to be* of Fano type *if there exists an effective, $\mathbb{Q}$-divisor $B$ such that $(X, B)$ is klt and weak log Fano pair. Equivalently: there exists a big $\mathbb{Q}$-divisor $B$ such that $K_X + B \sim_{\mathbb{Q}} 0$ and such that $(X, B)$ is a klt pair.*

*Remark 2.15* (Varieties of Fano type are Mori dream spaces)
If $X$ is of Fano type, recall from [BCHM10, Sect. 1.3] that $X$ is a "Mori dream space". Given any $\mathbb{R}$-Cartier divisor $D \in \mathbb{R}\,\mathrm{Div}(X)$, we can then run the $D$-Minimal Model Programme and obtain a sequence of extremal contractions and flips, $X \dashrightarrow Y$. If the push-forward of $D_Y$ of $Y$ is nef over, we call $Y$ a minimal model for $D$. Otherwise, there exists a $D_Y$-negative extremal contraction $Y \to T$ with $\dim Y > \dim T$, and we call $Y$ a Mori fibre space for $D$.

*Remark 2.16* (Relative notions). — As before, there exists an obvious relative version of the notion "Fano type". Remark 2.15 generalises to this relative setting.

Varieties of Fano type come in two flavours that often need to be treated differently. The following notion, which we recall for later use, has been introduced by Shokurov.

Definition 2.17 (Exceptional and non-exceptional pairs)
*Let $(X, B)$ be a projective pair, and assume that there exists an effective $P \in \mathbb{R}\,\mathrm{Div}(X)$ such that $K_X + B + P \sim_{\mathbb{R}} 0$. We say $(X, B)$ is* non-exceptional *if we can choose $P$ so that $(X, B + P)$ is* not *klt. We say that $(X, B)$ is* exceptional *if $(X, B + P)$ is klt for every choice of $P$.*

## 3. B-DIVISORS AND GENERALISED PAIRS

In addition to the classical notions for singularities of pairs that we recalled in Section 2.3 above, much of Birkar's work uses the notion of *generalised polarised pairs*. The additional flexibility of this notion allows for inductive proofs, but adds substantial technical difficulties. Generalised pairs were introduced by Birkar and Zhang in [BZ16].

*Disclaimer.* — The notion of generalised polarised pairs features prominently in Birkar's work, and should be presented in an adequate manner. The technical complications arising from this notion are however substantial and cannot be explained within a few pages. As a compromise, this section briefly explains what generalised pairs are, and how they come about in relevant settings. Section 4.4 pinpoints one place in Birkar's inductive scheme of proof where generalised pairs appear naturally, and explains why most (if not all) of the material presented in this survey should in fact be formulated and proven for generalised pairs. For the purpose of exposition, we will however ignore this difficulty and discuss the classical case only.

### 3.1. Definition of generalised pairs

To begin, we only recall a minimal subset of the relevant definitions, and refer to [Bir16a, Sect. 2] and to [BZ16, Sect. 4] for more details. We start with the notion of b-divisors, as introduced by Shokurov in [Sho96], in the simplest case.

Definition 3.1 (b-divisor). — *Let $X$ be a variety. We consider projective, birational morphisms $Y \to X$ from normal varieties $Y$, and for each $Y$ a divisor $M_Y \in \mathbb{R}\,\mathrm{Div}(Y)$. The collection $M := (M_Y)_Y$ is called* b-divisor *if for any morphism $f : Y' \to Y$ of birational models over $X$, we have $M_Y = f_*(M_{Y'})$.*

Definition 3.2 (b-$\mathbb{R}$-Cartier and b-Cartier b-divisors). — *Setting as in Definition 3.1. A b-divisor $M$ is called* b-$\mathbb{R}$-Cartier *if there exists one $Y$ such that $M_Y$ is $\mathbb{R}$-Cartier and such that for any morphism $f : Y' \to Y$ of birational models over $X$, we have $M_{Y'} = f^*(M_Y)$. Ditto for* b-Cartier *b-divisors.*

Definition 3.3 (Generalised polarised pair, [Bir16a, Sect. 2.13], [BZ16, Def. 1.4])
*Let $Z$ be a variety. A* generalised polarised pair *over $Z$ is a tuple consisting of the following data:*

(3.3.1) *a normal variety $X$ equipped with a projective morphism $X \to Z$,*
(3.3.2) *an effective $\mathbb{R}$-divisor $B \in \mathbb{R}\,\mathrm{Div}(X)$, and*
(3.3.3) *a b-$\mathbb{R}$-Cartier b-divisor over $X$ represented as $(\varphi : X' \to X, M')$, where $M' \in \mathbb{R}\,\mathrm{Div}(X')$ is nef over $Z$, and where $K_X + B + \varphi_* M'$ is $\mathbb{R}$-Cartier.*

*Notation 3.4* (Generalised polarised pair). — In the setup of Definition 3.3, we usually write $M := \varphi_* M'$ and say that $(X, B+M)$ is a generalised pair with data $X' \xrightarrow{\varphi} X \to Z$ and $M'$. In contexts where $Z$ is not relevant, we usually drop it from the notation: in this case one can just assume $X \to Z$ is the identity. When $Z$ is a point we also drop it but say the pair is projective.

*Observation 3.5.* — Following [BZ16, p. 286] we remark that Definition 3.3 is flexible with respect to $X'$ and $M'$. To be more precise, if $g : X'' \to X'$ is a projective birational morphism from a normal variety, then there is no harm in replacing $X'$ with $X''$ and replacing $M'$ with $g^* M'$.

### 3.2. Singularities of generalised pairs

All notions introduced in Section 2.3 have analogues in the setting of generalised pairs. Again, we cover only the most basic definition here.

Definition 3.6 (Generalised log discrepancy, singularity classes)
*Consider a generalised polarised pair $(X, B + M)$ with data $X' \xrightarrow{\varphi} X \to Z$ and $M'$, where $\varphi$ is a log resolution of $(X, B)$. Then, there exists a uniquely determined divisor $B'$ on $X'$ such that*

$$K_{X'} + B' + M' = \varphi^*(K_X + B + M)$$

*If $D \in \mathrm{Div}(X')$ is any prime divisor, the* generalised log discrepancy *is defined to be*

$$a_{\log}(D, X, B + M) := 1 - \mathrm{mult}_D\, B'.$$

*As before, we define the* generalised total log discrepancy $a_{\log}(X, B + M)$ *by taking the infimum over all $D$ and all resolutions. In analogy to the definitions of Table 1, we say that the generalised polarised pair is* generalised lc *if $a_{\log}(X, B + M) \geq 0$. Ditto for all the other definitions.*

### 3.3. Example: Fibrations and the canonical bundle formula

We discuss a setting where generalised pairs appear naturally. Let $Y$ be a normal pair variety, and let $f : Y \to X$ be a fibration: the space $X$ is projective, normal and of positive dimension, the morphism $f$ is surjective with connected fibres. Also, assume that $K_Y$ is $\mathbb{Q}$-linearly equivalent to zero over $X$, so that there exists $L_X \in \mathbb{Q}\,\mathrm{Div}(X)$ with $K_Y \sim_{\mathbb{Q}} f^* L_X$. Ideally, one might hope that it would be possible to choose $L_X = K_X$, but this is almost always wrong — compare Kodaira's formula for the canonical bundle of an elliptic fibration, [BHPVdV04, Sect. V.12]. To fix this issue, we define a first correction term $B \in \mathbb{Q}\,\mathrm{Div}(X)$ as

$$B := \sum_{\substack{D \in \mathrm{Div}(X) \\ \text{prime}}} (1 - t_D) \cdot D \quad \text{where} \quad t_D := \mathrm{lct}^\circ\big(Y,\, \Delta_Y,\, f^* D\big)$$

The symbol $\mathrm{lct}^\circ$ denotes a variant of the lc threshold introduced in Definition 2.9, which measures the singularities of $\big(Y, f^* D\big)$ only over the generic point of $D$. Since $X$ is smooth in codimension one, this also solves the problem of defining $f^* D$. Finally, one chooses $M \in \mathbb{Q}\,\mathrm{Div}(X)$ such that $K_X + B + M$ is $\mathbb{Q}$-Cartier and such that the desired $\mathbb{Q}$-linear equivalence holds,

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + B + M).$$

The divisor $B$ is usually called the "discriminant part" of the correction term. It detects singularities of the fibration, such as multiple or otherwise singular fibres, over codimension one points of $X$. The divisor $M$ is called the "moduli part". It is harder to describe. While we have defined it only up to $\mathbb{Q}$-linear equivalence, a more involved construction can be used to define it as an honest divisor.

*Commentary.* — Conjecturally, the moduli part carries information on the birational variation of the fibres of $f$, [Kaw98]. We refer to [Kol07] and to the introduction of the recent research paper [FL18] for an overview, but see also [FG14].

**3.3.1.** *Behaviour under birational modifications.* — We ask how the moduli part of the correction term behaves under birational modification. To this end, let $\varphi : X' \to X$ be a birational morphism of normal, projective varieties. Choosing a resolution $Y'$ of $Y \times_X X'$, we find a diagram as follows,

$$\begin{tikzcd} Y' \arrow[r, "\Phi\text{, birational}"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "\varphi\text{, birational}"'] & X. \end{tikzcd}$$

Set $\Delta_{Y'} := \Phi^* K_Y - K_{Y'}$. Generalising the definition of $\operatorname{lct}^\circ$ a little to allow for negative coefficients in $\Delta_{Y'}$, one can then define $B'$ similarly to the construction above,

$$B' := \sum_{\substack{D \in \operatorname{Div}(X') \\ \text{prime}}} (1 - t'_D) \cdot D \quad \text{where} \quad t'_D := \operatorname{lct}^\circ\big(Y',\, \Delta_{Y'},\, (f')^* D\big).$$

Finally, one may then choose $M' \in \mathbb{Q}\operatorname{Div}(X')$ such that

$$K_{Y'} + \Delta_{Y'} \sim_{\mathbb{Q}} (f')^*(K_{X'} + B' + M'),$$
$$K_{X'} + B' + M' = \varphi^*(K_X + B + M)$$

and $B = \varphi_* B'$ as well as $M = \varphi_* M'$.

**3.3.2.** *Relation to generalised pairs.* — Now assume that $Y$ is lc. The divisor $B$ will then be effective. However, much more is true: after passing to a certain birational model $X'$ of $X$, the divisor $M_{X'}$ is nef and for any higher birational model $X'' \to X'$, the induced $M_{X''}$ on $X''$ is the pullback of $M_{X''}$, [Kaw98, Amb04, Kol07] and summarised in [Bir16a, Thm. 3.6]. In other words, going to a sufficiently high birational model of $X'$ of $X$, the moduli parts $M'$ define an b-$\mathbb{R}$-Cartier b-divisor. Moreover, this b-divisor is b-nef. We obtain a generalised polarised pair $(X, B + M)$ with data $X' \xrightarrow{\varphi} X \to \operatorname{Spec} \mathbb{C}$ and $M'$. This generalised pair is generalised lc by definition.

*Commentary.* — A famous conjecture of Prokhorov and Shokurov [PS09, Conj. 7.13] asserts that the moduli divisor $M_{X''}$ is semiample, on any sufficiently high birational model $X''$ of $X$. More precisely, it is expected that a number $m$ exists that depends only on the general fibre of $f$ such that all divisors $m \cdot M_{X''}$ are basepoint free. If this conjecture was solved, it is conceivable that Birkar's work could perhaps be rewritten in a manner that avoids the notion of generalised pairs.

*Remark 3.7* (Outlook). — The construction outlined in this section is used in the proof of "Boundedness of complements", as sketched in Section 4.4 below. It generalises fairly directly to pairs $(Y, \Delta_Y)$, and even to tuples where $\Delta_Y$ is not necessarily effective, [Bir16a, Sect. 3.4].

## 4. BOUNDEDNESS OF COMPLEMENTS

### 4.1. Statement of result

One of the central concepts in Birkar's papers [Bir16a, Bir16b] is that of a *complement*. The notion of a "complement" is an ingenious concept of Shokurov that was introduced in his investigation of threefold flips, [Sho92, Sect. 5]. We recall the definition in brief.

DEFINITION 4.1 (Complement, [Bir16a, Sect. 2.18]). — *Let $(X, B)$ be a projective pair and $m \in \mathbb{N}$. An $m$-*complement *of $K_X + B$ is a $\mathbb{Q}$-divisor $B^+$ with the following properties.*

(4.1.1) *The tuple $(X, B^+)$ is an lc pair.*
(4.1.2) *The divisor $m \cdot (K_X + B^+)$ is linearly equivalent to $0$. In particular, $m \cdot B^+$ is integral.*
(4.1.3) *We have $m \cdot B^+ \geq m \cdot \lfloor B \rfloor + \lfloor (m+1) \cdot \{B\} \rfloor$.*

*Remark 4.2* (Complements give sections). — Setting as in Definition 4.1. If $m$ can be chosen such that $m \cdot \lfloor B \rfloor + \lfloor (m+1) \cdot \{B\} \rfloor \geq m \cdot B$, then Item (4.1.2) guarantees that $-m \cdot (K_X + B)$ is linearly equivalent to the effective divisor $m \cdot (B^+ - B)$. In particular, the sheaf $\mathscr{O}_X\big(-m \cdot (K_X + B)\big)$ admits a global section.

*Remark 4.3.* — In view of Item (4.1.2), Shokurov considers complements as divisors that make the lc pair $(X, B^+)$ "Calabi–Yau", hence "flat".

The following result, which asserts the existence of complements with bounded $m$, is one of the core results in Birkar's paper [Bir16a]. A proof of Theorem 4.4 is sketched in Section 4.4 on the next page.

THEOREM 4.4 (Boundedness of complements, [Bir16a, Thm. 1.7])
*Given $d \in \mathbb{N}$ and a finite set $\mathcal{R} \subset [0, 1] \cap \mathbb{Q}$, there exists $m \in \mathbb{N}$ with the following property. If $(X, B)$ is any log canonical, projective pair, where*

(4.4.1) *$X$ is of Fano type and $\dim X = d$,*
(4.4.2) *the coefficients of $B$ are of the form $\frac{\ell - r}{\ell}$, for $r \in \mathcal{R}$ and $\ell \in \mathbb{N}$,*
(4.4.3) *$-(K_X + B)$ is nef,*

*then there exists an $m$-complement $B^+$ of $K_X + B$ that satisfies $B^+ \geq B$. The divisor $B^+$ is also an $(m \cdot \ell)$-complement, for every $\ell \in \mathbb{N}$.*

*Remark 4.5* (Complements give sections). — Given a pair $(X, B)$ as in Theorem 4.4 and a number $\ell \in \mathbb{N}$ such that $(m\ell) \cdot B$ is integral, then $m\ell \cdot \lfloor B \rfloor + \lfloor (m\ell + 1) \cdot \{B\} \rfloor \geq m\ell \cdot B$, and Remark 4.2 implies that $h^0\big(X,\, \mathscr{O}_X(-m\ell \cdot (K_X + B))\big) > 0$.

### 4.2. Idea of application

We aim to show Theorem 1.2: under suitable assumptions on the singularities the family of Fano varieties is bounded. The proof relies on the following boundedness criterion of Hacon and Xu that we quote without proof (but see Sections 1.1.1 and 1.1.2 for a brief discussion). Recall that a set of numbers is DCC if every strictly descending sequence of elements eventually terminates.

THEOREM 4.6 (Boundedness criterion, [HX15, Thm. 1.3]). — *Given $d \in \mathbb{N}$ and a* DCC *set $I \subset [0,1] \cap \mathbb{Q}$, let $\mathcal{Y}_{d,I}$ be the family of pairs $(X, B)$ such that the following holds true.*

(4.6.1) *The pair $(X, B)$ is projective, klt, and of dimension $\dim_{\mathbb{C}} X = d$.*

(4.6.2) *The coefficients of $B$ are contained in $I$. The divisor $B$ is big and $K_X + B \sim_{\mathbb{Q}} 0$.*

*Then, the family $\mathcal{Y}_{d,I}$ is bounded.* □

With the boundedness criterion in place, the following observation relates "boundedness of complements" to "boundedness of Fanos" and explains what pieces are missing in order to obtain a full proof.

*Observation 4.7.* — Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, Theorem 4.4 gives a number $m \in \mathbb{N}$ such that every $\varepsilon$-lc Fano variety $X$ with $-K_X$ nef admits an effective complement $B^+$ of $K_X = K_X + 0$, with coefficients in the set $\{\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}\}$. If one could in addition always choose $B^+$ so that $(X, B^+)$ was klt rather than merely lc, then Theorem 4.6 would immediately apply to show that the family of $\varepsilon$-lc Fano varieties with $-K_X$ nef is bounded.

As an important step towards boundedness of $\varepsilon$-lc Fanos, we will see in Section 5 how the theorem on "effective birationality" together with Theorem 4.6 and Observation 4.7 can be used to find a boundedness criterion (=Proposition 5.3 on page 17) that applies to a relevant class of klt, weak Fano varieties.

### 4.3. Variants and generalisations

Theorem 4.4 is in fact part of a much larger package, including boundedness of complements in the relative setting, [Bir16a, Thm. 1.8], and boundedness of complements for generalised polarised pairs, [Bir16a, Thm. 1.10]. To keep this survey reasonably short, we do not discuss these results here, even though they are of independent interest, and play a role in the proofs of Theorems 4.4 and 1.2.

### 4.4. Idea of proof for Theorem 4.4

We sketch a proof of "boundedness of complements", following [Bir16a, p. 6ff] in broad strokes, and filling in some details now and then. In essence, the proof works by induction over the dimension, so assume that $d$ is given and that everything was already shown for varieties of lower dimension.

*Simplification.* — Theorem 4.4 considers a finite set $\mathcal{R} \subset [0,1] \cap \mathbb{Q}$, and log canonical pairs $(X, B)$, where the coefficients of $B$ are contained in the set

$$\Phi(\mathcal{R}) := \left\{ \tfrac{\ell - r}{\ell} \,|\, r \in \mathcal{R} \text{ and } \ell \in \mathbb{N} \right\}.$$

The set $\Phi(\mathcal{R})$ is infinite, and has $1 \in \mathbb{Q}$ as its only accumulation point. Birkar shows that it suffices to treat the case where the coefficient set is finite. To this end, he constructs in [Bir16a, Prop. 2.49 and Constr. 6.13] a number $\varepsilon' \ll 1$ and shows that it suffices to consider pairs with coefficients in the finite set $\Phi(\mathcal{R}) \cap [0, 1-\varepsilon'] \cup \{1\}$. In fact, given any $(X, B)$, he considers the divisor $B'$ obtained by replacing those coefficients on $B$ that lie in the range $(1-\varepsilon', 1)$ with 1. Next, he constructs a birational model $(X'', B'')$ of $(X, B')$ that satisfies all assumptions Theorem 4.4. His construction guarantees that to find an $n$-complement for $(X, B)$ it is equivalent to find an $n$-complement for $(X'', B'')$. Among other things, the proof involves carefully constructed runs of the Minimal Model Programme, Hacon–M$^{\text{c}}$Kernan–Xu's local and global ACC for log canonical thresholds [HMX14, Thms. 1.1 and 1.5], and the extension of these results to generalised pairs [BZ16, Thm. 1.5 and 1.6].

*Remark 4.8.* — Recall from Remark 2.15 that Assumption (4.4.1) ("$X$ is of Fano type") allows us to run Minimal Model Programmes on arbitrary divisors.

Along similar lines, Birkar is able to modify $(X'', B'')$ by further birational transformation, and eventually proves that it suffices to show boundedness of complements for pairs that satisfy the following additional assumptions.

*Assumption 4.9.* — The coefficient set of $(X, B)$ is contained in $\mathcal{R}$ rather than in $\Phi(\mathcal{R})$, and one of the following holds true.

(4.9.1) The divisor $-(K_X + B)$ is nef and big, and $B$ has a component $S$ with coefficient 1 that is of Fano type.

(4.9.2) There exists a fibration $f\colon X \to T$ and $K_X + B \equiv 0$ along that fibration.

(4.9.3) The pair $(X, B)$ is exceptional.

*Commentary.* — The main distinction is between Case (4.9.3) and Case (4.9.1). In fact, if $(X, B)$ is not exceptional, recall from Definition 2.17 that there exists an effective $P \in \mathbb{R}\,\mathrm{Div}(X)$ such that $K_X + B + P \sim_{\mathbb{R}} 0$ and such that $(X, B + P)$ is *not* klt. This allows us to find a birational model whose boundary contains a divisor with multiplicity one. Case (4.9.2) comes up if the runs of the Minimal Model Programmes used in the construction of birational models terminates with a Kodaira fibre space.

The three cases (4.9.1)–(4.9.3) require very different inductive treatments.

*Case* (4.9.1). — We consider only the simple case where $S = \lfloor B \rfloor$ is a normal prime divisor, where $(X, B)$ is plt near $S$ and where $-(K_X + B)$ is ample. Setting $B_S := (K_X + B)|_S - K_S$, the coefficients are contained in a finite set $\mathcal{R}'$ of rational numbers that depends only on $\mathcal{R}$ and on $d$. In summary, the pair $(S, B_S)$ reproduces the assumptions of Theorem 4.4, and by induction we obtain a number $n \in \mathbb{N}$ that depends only on $\mathcal{R}$ and $d$, such that

(4.9.4) the divisor $n \cdot B_S$ is integral, and

(4.9.5) there exists an $n$-complement $B_S^+$ of $K_S + B_S$.

Following [Bir16a, Prop. 6.7], we aim to extend $B_S^+$ from $S$ to a complement $B^+$ of $K_X + B$ on $X$. As we saw in in Remark 4.5, Item (4.9.4) guarantees that $n \cdot (B_S^+ - B_S)$ is effective, so that the complement $B_S^+$ gives rise to a section in

$$H^0\big(S,\, n \cdot (B_S^+ - B_S)\big) = H^0\big(S,\, -n \cdot (K_S + B_S)\big)$$

But then, looking at the cohomology of the standard ideal sheaf sequence,

$$H^0\big(X,\, -n \cdot (K_X + B)\big) \to \underbrace{H^0\big(S,\, -n \cdot (K_X + B)|_S\big)}_{\neq 0 \text{ by Rem. } 4.5} \to \underbrace{H^1\big(X,\, -n \cdot (K_X + B) - S\big)}_{=0 \text{ by Kawamata–Viehweg vanishing}}$$

we find that the section extends to $X$ and defines an associated divisor $B^+ \in |-(K_X + B)|_{\mathbb{Q}}$. Using the connectedness principle for non-klt centres[3], one argues that $B^+$ is the desired complement.

*Case* (4.9.2). — Given a fibration $f : X \to T$, we apply the construction of Section 3.3, in order to equip the base variety $T$ with the structure of a generalised polarised pair $(T, B + M)$, with data $T' \xrightarrow{\varphi} T \to \operatorname{Spec} \mathbb{C}$ and $M'$.

Adding to the results explained in Section 3.3, Birkar shows that the coefficients of $B$ and $M$ are not arbitrary. The coefficients of $B$ are in $\Phi(\mathcal{S})$ for some fixed finite set $\mathcal{S}$ of rational numbers that depends only on $\mathcal{R}$ and $d$. Along similar lines, there exists a bounded number $p \in \mathbb{N}$ such that $p \cdot M$ is integral. The plan is now to use induction to find a bounded complement for $K_T + B + M$ and pull it back to $X$. This plan works out well, but requires us to formulate and prove all results pertaining to boundedness of complements in the setting of generalised polarised pairs. All the arguments sketched here continue to work, *mutatis mutandis*, but the level of technical difficulty increases substantially.

*Case* (4.9.3). — There is little that we can say in brief about this case. Still, assume for simplicity that $B = 0$ and that $X$ is a Fano variety. If we could show that $X$ belongs to a bounded family, then we would be done. Actually we need something weaker: effective birationality. Assume we have already proved Theorem 5.1. Then there is a bounded number $m \in \mathbb{N}$ such that $|-mK_X|$ defines a birational map. Pick $M \in |-mK_X|$ and let $B^+ := \frac{1}{m} \cdot M$. Since $X$ is exceptional, $(X, B^+)$ is automatically klt, hence $K_X + B^+$ is an $m$-complement.

[3] For generalised pairs, this is [Bir16a, Lem. 2.14]

Although this gives some idea of how one may get a bounded complement but in practice we cannot give a complete proof of Theorem 5.1 before proving Theorem 4.4. Contrary to the exposition of this survey paper, where "boundedness of complements" and "effective birationality" are treated as if they were separate, the proofs of the two theorems are in fact much intertwined, and this is one of the main points where they come together. Many of the results discussed in this overview ("Bound on anti-canonical volumes", "Bound on lc thresholds") have separate proofs in the exceptional case.

## 5. EFFECTIVE BIRATIONALITY

### 5.1. Statement of result

The second main ingredient in Birkar's proof of boundedness is the following result. A proof is sketched in Section 4.4 on page 14.

THEOREM 5.1 (Effective birationality, [Bir16a, Thm. 1.2]). — *Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, there exists $m \in \mathbb{N}$ with the following property. If $X$ is any $\varepsilon$-lc weak Fano variety of dimension $d$, then $|-m \cdot K_X|$ defines a birational map.*

*Remark 5.2.* — The divisors $m \cdot K_X$ in Theorem 5.1 need not be Cartier. The linear system $|-m \cdot K_X|$ is the space of effective Weil divisors on $X$ that are linearly equivalent to $-m \cdot K_X$.

### 5.2. Idea of application

In the framework of [Bir16a], effective birationality is used to improve the boundedness criterion spelled out in Theorem 4.6 above.

PROPOSITION 5.3 (Boundedness criterion, [Bir16a, Prop. 7.13])

*Let $d, v \in \mathbb{N}$ and let $(t_\ell)_{\ell \in \mathbb{N}}$ be a sequence of positive real numbers. Let $\mathcal{X}$ be the family of projective varieties $X$ with the following properties.*

(5.3.1) *The variety $X$ is a klt weak Fano variety of dimension $d$.*

(5.3.2) *The volume of the canonical class is bounded, $\operatorname{vol}(-K_X) \leq v$.*

(5.3.3) *For every $\ell \in \mathbb{N}$ and every $L \in |-\ell \cdot K_X|$, the pair $(X, t_\ell \cdot L)$ is klt.*

*Then, $\mathcal{X}$ is a bounded family.*

*Remark 5.4.* — The formulation of Proposition 5.3 is meant to illustrate the application of Theorem 5.1 to the boundedness problem. It is a simplified version of Birkar's formulation and defies the logic of his work. While we present Proposition 5.3 as a corollary to Theorem 5.1, and to all the results mentioned in Section 4, Birkar uses [Bir16a, Prop. 7.13] as one step in the inductive proof of "boundedness of complements" and "effective birationality". That requires him to explicitly list partial cases of "boundedness of complements" and "effective birationality" as assumptions to the proposition, and makes the formulation more involved.

*Remark 5.5.* — Proposition 5.3 reduces the boundedness problem to solving the following two problems.

- Boundedness of volumes, as required in (5.3.2). This is covered in the subsequent Section 6.
- Existence of numbers $t_\ell$, as required in (5.3.3). This amounts to bounding "lc thresholds" and is covered in Section 7.

To prove Proposition 5.3, Birkar uses effective birationality in the following form, as a log birational boundedness result.

PROPOSITION 5.6 (Log birational boundedness of certain pairs, [Bir16a, Prop. 4.4]) *Given $d, v \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$. Then, there exists $c \in \mathbb{R}^+$ and a bounded family $\mathcal{P}$ of couples with the following property. If $X$ is a normal projective variety of dimension $d$ and if $B \in \mathbb{R}\operatorname{Div}(X)$ and $M \in \mathbb{Q}\operatorname{Div}(X)$ are divisors such that the following holds,*

(5.6.1) *the divisor $B$ is effective, with coefficients in $\{0\} \cup [\varepsilon, \infty)$,*
(5.6.2) *the divisor $M$ is effective, nef and $|M|$ defines a birational map,*
(5.6.3) *the difference $M - (K_X + B)$ is pseudo-effective,*
(5.6.4) *the volume of $M$ is bounded, $\operatorname{vol}(M) < v$,*
(5.6.5) *if $D$ is any component of $M$, then $\operatorname{mult}_D(B + M) \geq 1$,*

*then there exists a log smooth couple $(X', \Sigma) \in \mathcal{P}$, a rational map $\overline{X} \dashrightarrow X$ and a resolution of singularities $r : \widetilde{X} \to X$, with the following properties.*

(5.6.6) *The divisor $\Sigma$ contains the birational transform on $M$, as well as the exceptional divisor of the birational map $\beta$.*
(5.6.7) *The movable part $A_{\widetilde{X}}$ of $r^*M$ is basepoint free.*
(5.6.8) *If $\widetilde{X}'$ is any resolution of $X$ that factors via $X'$ and $\widetilde{X}$,*

$$\begin{tikzcd}
\widetilde{X}' \arrow[r, "\widetilde{\beta},\ \text{birational}"] \arrow[d, "s,\ \text{resolution}"'] & \widetilde{X} \arrow[d, "r,\ \text{resolution}"] \\
X' \arrow[r, dashed, "\beta,\ \text{birational}"'] & X
\end{tikzcd}$$

*then the coefficients of the $\mathbb{Q}$-divisor $s_*(r \circ \widetilde{\beta})^*M$ are at most $c$ and $\widetilde{\beta}^* A_{\widetilde{X}}$ is linearly equivalent to zero relative to $X'$.*

*Sketch of proof for Proposition 5.6, following* [Bir16a, p. 42]. — Since $|M|$ defines a birational map, there exists a resolution $r : \widetilde{X} \to X$ such that $r^*M$ decomposes as the sum of a base point free movable part $A_{\widetilde{X}}$ and fixed part $R_{\widetilde{X}}$. The contraction $X \to X''$ defined by $A_{\widetilde{X}}$ is birational. Since $\operatorname{vol}(M)$ is bounded, the varieties $X''$ obtained in this way are all members of one bounded family $\mathcal{P}'$. The family $\mathcal{P}'$ is however not yet the desired family $\mathcal{P}$, and the varieties in $\mathcal{P}'$ are not yet equipped with an appropriate boundary. To this end, one needs to invoke a criterion of Hacon–McKernan–Xu for "log birationally boundedness", [HMX13, Lem. 2.4.2(4)], and take an appropriate resolution of the elements in $\mathcal{P}'$. □

*Sketch of proof for Proposition 5.3, following* [Bir16a, p. 80]. — Applying Theorems 4.4 ("Boundedness of complements") and 5.1 ("Effective birationality"), we find a number $m \in \mathbb{N}$ such that every $X \in \mathcal{X}$ admits an $m$-complement for $K_X$ and that $|-m \cdot K_X|$ defines a birational map. If $m$-complements $B^+$ of $K_X$ could always be chosen such that $(X, B^+)$ were klt, we have seen in Observation 4.7 that $\mathcal{X}$ is bounded. However, Theorem 4.4 guarantees only the existence of an $m$-complement $B^+$ of $K_X$ where $(X, B^+)$ is lc. Using the bounded family $\mathcal{P}$ obtained when applying Proposition 5.6 with $M = -m \cdot K_X$ and $B = 0$, we aim to find a universal constant $\ell$ and a finite set $\mathcal{R}$, and then perturb any given $(X, B^+)$ in order to find a boundary $B^{++}$ with coefficients in $\mathcal{R}$ that is $\mathbb{Q}$-linearly equivalent to $-K_X$ and makes $(X, B^{++})$ klt. Boundedness will then again follow from Theorem 4.6.

To spell out a few more details of the proof use boundedness of the family $\mathcal{P}$ to infer the existence of a universal constant $\ell$ with the following property.

> If $(X', \Sigma) \in \mathcal{P}$ and if $A_{X'} \in \mathrm{Div}(X')$ is contained in $\Sigma$ with coefficients bounded by $c$, and if $|A_{X'}|$ is basepoint free and defines a birational morphism, then there exists $G_{X'} \in |\ell \cdot A_{X'}|$ whose support contains $\Sigma$.

Now assume that one $X \in \mathcal{X}$ is given. It suffices to consider the case where $X$ is $\mathbb{Q}$-factorial and admits an $m$-complement of the form $B^+ = \frac{1}{m} \cdot M$, for general $M \in |-m \cdot K_X|$. To make use of $\ell$, consider a diagram as discussed in Item (5.6.8) of Proposition 5.6 above and decompose $r^*M = A_{\widetilde{X}} + R_{\widetilde{X}}$ into its moving and its fixed part. Write $A := r_* A_{\widetilde{X}}$ and $R := r_* R_{\widetilde{X}}$. Item (5.6.6) of Proposition 5.6 implies that the divisor $A_{X'} := s_* \widetilde{\beta}^* A_{\widetilde{X}}$ is then contained in $\Sigma$, and Item (5.6.8) asserts that it is basepoint free, defines a birational morphism. So, we find $G_{X'} \in |\ell \cdot A_{X'}|$ as above. Writing $G := r_* \widetilde{\beta}_* s^* G_{X'}$, we find that $G + \ell \cdot R \in |-m\ell \cdot K_X|$, so that $(X, t_{m\ell} G)$ is klt by assumption. We may assume that $t_{m\ell}$ is rational and $t_{m\ell} < \frac{1}{m\ell}$. If $(X, \frac{1}{m\ell}(G + \ell \cdot R))$ is lc, then set $B' := \frac{1}{m\ell}(G + \ell \cdot R)$. Otherwise, one needs to use the lower-dimensional versions of the variants and generalisations of boundedness of complements that we discussed in Section 4.3 above. To be more precise, using

(5.6.9) boundedness of complements for generalised polarised pairs for varieties of dimension $\leq d - 1$, and

(5.6.10) boundedness of complements in the relative setting for varieties of dimension $d$,

one can always find a universal number $n$ and $B' \geq t_{m\ell} \cdot (G + \ell \cdot R)$ where $(X, B')$ is lc and $n \cdot (K_X + B') \sim 0$. Finally, set

$$B^{++} := \frac{1}{2} \cdot B^+ + \frac{t}{2m} \cdot A - \frac{t}{2m\ell} \cdot G + \frac{1}{2} \cdot B'$$

and then show by direct computation that all required properties hold. $\square$

### 5.3. Preparation for the proof of Theorem 5.1

We prepare for the proof with the following proposition. In essence, it asserts that effective divisors with "degree" bounded from above cannot have too small lc thresholds, under appropriate assumptions. Since this proposition may look plausible, we do not go into details of the proof. Further below, Proposition 7.3 gives a substantially stronger result whose proof is sketched in some detail.

Proposition 5.7 (Singularities in bounded families, [Bir16a, Prop. 4.2])
*Given $\varepsilon' \in \mathbb{R}^+$ and given a bounded family $\mathcal{P}$ of couples, there exists a number $\delta \in \mathbb{R}^{>0}$ such that the following holds. Given the the following data,*

(5.7.1) *an $\varepsilon'$-lc, projective pair $(\widehat{G}, \widehat{B})$,*
(5.7.2) *a reduced divisor $T \in \operatorname{Div}(\widehat{G})$ such that $\big(\widehat{G}, \operatorname{supp}(\widehat{B}+T)\big) \in \mathcal{P}$, and*
(5.7.3) *an $\mathbb{R}$-divisor $\widehat{N}$ whose support is contained in $T$, and whose coefficients have absolute values $\leq \delta$,*

*then $(\widehat{G}, \widehat{B}+\widehat{L})$ is klt, for all $\widehat{L} \in |\widehat{N}|_{\mathbb{R}}$.* □

### 5.4. Sketch of proof of Theorem 5.1

Assume that numbers $d$ and $\varepsilon$ are given. Given an $\varepsilon$-lc Fano variety $X$ of dimension $d$, we will be interested in the following two main invariants,

$$m_X := \min\{\, m' \in \mathbb{N} \,|\, \text{the linear system } |-m' \cdot K_X| \text{ defines a birational map } \}$$
$$n_X := \min\{\, n' \in \mathbb{N} \mid \operatorname{vol}(-n' \cdot K_X) \geq (2d)^d \,\}$$

Eventually, it will turn out that both numbers are bounded from above. Our aim here is to bound the numbers $m_X$ by a constant that depends only on $d$ and $\varepsilon$.

#### Bounding the quotient

Following [Bir16a], we will first find an upper bound for the quotients $m_X/n_X$ by a number that depends only on $d$ and $\varepsilon$.

**5.4.1.** *Construction of non-klt centres.* — In the situation at hand, a standard method ("tie breaking") allows us to find dominating families of non-klt centres; we refer to [Kol97, Sect. 6] for an elementary discussion, but see also [Bir16a, Sect. 2.31]. Given an $\varepsilon$-lc Fano variety $X$ of dimension $d$, and using the assumption that $\operatorname{vol}(-n_X \cdot K_X) \geq (2d)^d$, the following has been shown by Hacon, McKernan and Xu.

*Claim 5.8* (Dominating family of non-klt centres, [HMX14, Lem. 7.1])
Given any $\varepsilon$-lc Fano variety $X$, there exists a dominating family $\mathcal{G}_X$ of subvarieties in $X$ with the following property. If $(x, y) \in X \times X$ is any general tuple of points, then there exists a divisor $\Delta \in |-(n_X+1) \cdot K_X|_{\mathbb{R}}$ such that the following holds.

(5.8.1) The pair $(X, \Delta)$ is not klt at $y$.
(5.8.2) The pair $(X, \Delta)$ is lc near $x$ with a unique non-klt place. The associated non-klt centre is a subvariety of the family $\mathcal{G}_X$. □

Given $X$, we may assume that the members of the families $\mathcal{G}_X$ all have the same dimension, and that this dimension is minimal among all families of subvarieties that satisfy (5.8.1) and (5.8.2).

**5.4.2.** *The case of isolated centres.* — If $X$ is given such that the members of $\mathcal{G}_X$ are points, then the elements are isolated non-klt centres. Given $G \in \mathcal{G}_X$, standard vanishing theorems for multiplier ideals will then show surjectivity of the restriction maps

$$H^0\big(X,\, \mathscr{O}_X(K_X + \Delta)\big) \to \underbrace{H^0\big(G,\, \mathscr{O}_X(K_X + \Delta)|_G\big)}_{\cong \mathbb{C}}.$$

In particular, we find that $\mathscr{O}_X(K_X + \Delta) \cong \mathscr{O}_X(-n_X \cdot K_X)$ has non-trivial sections. Further investigation reveals that a bounded multiple of $-n_X \cdot K_X$ will in fact give a birational map.

**5.4.3.** *Non-isolated centres.* — It remains to consider varieties $X$ where the members of $\mathcal{G}_X$ are positive-dimensional. Following [Bir16a, proofs of Prop. 4.6 and 4.8], we trace the arguments for that case in *very* rough strokes, ignoring all of the (many) subtleties along the way. The main observation to handle this case is the following volume bound.

*Claim 5.9* (Volume bound, [Bir16a, Step 3 on p. 48]). — There exists a number $v \in \mathbb{R}^+$ that depends only on $d$ and $\varepsilon$, such that for all $X$ and all positive-dimensional $G \in \mathcal{G}_X$, we have $\operatorname{vol}(-m_X \cdot K_X|_G) < v$.

*Idea of proof for Claim 5.9.* — Going back and looking at the construction of non-klt centres (that is, the detailed proof of Claim 5.8), one finds that the construction can be improved to provide families of lower-dimension centres if only the volumes are big enough. But this collides with our assumption that the varieties in $\mathcal{G}_X$ were of minimal dimension. □ (Claim 5.9)

To make use of Claim 5.9, look at one $X$ where the members of $\mathcal{G}_X$ are positive-dimensional. Choose a general divisor[(4)] $M \in |-m_X \cdot K_X|$, and let $(x, y) \in X \times X$ be a general tuple of points with associated centre $G \in \mathcal{G}_X$. Since $G$ is a non-klt centre that has a unique place over it, adjunction (and inversion of adjunction) works rather well. Together with the bound on volumes, this allows us to define a natural boundary $\widehat{B}$ on a suitable birational modification $\widehat{G}$ of the normalisation of $G$, such that the following holds.

(5.10.1) The pair $(\widehat{G}, \widehat{B})$ is $\varepsilon'$-lc, for some controllable number $\varepsilon'$.

(5.10.2) Writing $E$ for the exceptional divisor of $\widehat{G} \to G$ and $T := (\widehat{B} + E)_{\mathrm{red}}$, the couple $\big(\widehat{G}, \operatorname{supp}(\widehat{B} + T)\big)$ belongs to a bounded family $\mathcal{P}$ that in turn depends only on the numbers $d$ and $\varepsilon$.

(5.10.3) The pull-back of $M$ to $\widehat{G}$ has support in $\operatorname{supp}(\widehat{B} + T)$.

(4) the divisor $M$ should really be taken as the movable part, but we ignore this detail.

**5.4.4.** *End of proof.* — The idea now is of course to apply Proposition 5.7, using the family $\mathcal{P}$. Arguing by contradiction, we assume that the numbers $m_X/n_X$ are unbounded. We can then find one $X$ where $n_X/m_X$ is really quite small when compared to the number $\delta$ given by Proposition 5.7. In fact, taking $\widehat{N}$ as the pull-back of $\frac{n_X}{m_X} \cdot M$, it is possible to guarantee that the coefficients of $\widehat{N}$ are smaller than $\delta$.

Intertwining this proof with the proof of "boundedness of complements", we may use a partial result from that proof, and find $L \in |-n_X \cdot K_X|_{\mathbb{Q}}$, whose coefficients are $\geq 1$. Since the points $(x, y) \in X \times X$ were chosen generically, the pull-back $\widehat{L}$ of $L$ to $\widehat{G}$ has coefficients $\geq 1$, and can therefore never appear in the boundary of a klt pair. But then, $\widehat{L} \in |\widehat{N}|_{\mathbb{R}}$, which contradicts Proposition 5.7 and ends the proof. In summary, we were able to bound the quotient $m_X/n_X$ by a constant that depends only on $d$ and $\varepsilon$. □ (Boundedness of quotients)

### Bounding the numbers $m_X$

Finally, we still need to bound $m_X$. This can be done by arguing that the volumes $\mathrm{vol}(-m_X \cdot K_X)$ are bounded from above, and then use the same set of ideas discussed above, using $X$ instead of a birational model $\widehat{G}$ of its subvariety $G$. Since some of the core ideas that go into boundedness of volumes are discussed in more detail in the following Section 6 below, we do not go into any details here. □

## 6. BOUNDS FOR VOLUMES

### 6.1. Statement of result

Once Theorem 1.2 ("Boundedness of Fanos") is shown, the volumes of anticanonical divisors of $\varepsilon$-lc Fano varieties of any given dimension will clearly be bounded. Here, we discuss a weaker result, proving boundedness of volumes for Fanos of dimension $d$, assuming boundedness of Fanos in dimension $d - 1$.

Theorem 6.1 (Bound on volumes, [Bir16a, Thm. 1.6]). — *Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, if the $\varepsilon$-lc Fano varieties of dimension $d - 1$ form a bounded family, then there is a number $v$ such that $\mathrm{vol}(-K_X) \leq v$, for all $\varepsilon$-lc weak Fano varieties $X$ of dimension $d$*

### 6.2. Idea of application

We have seen in Section 5.2 how to obtain boundedness criteria for families of varieties from boundedness of volumes. This makes Theorem 6.1 a key step in the inductive proof of Theorem 1.2.

### 6.3. Idea of proof for boundedness of volumes, following [Bir16a, Sect. 9]

To illustrate the core idea of proof, we consider only the simplest cases and make numerous simplifying assumptions, no matter how unrealistic. The assumption that $\varepsilon$-lc Fano varieties of dimension $d-1$ form a bounded family will be used in the following form.

LEMMA 6.2 (Consequence of boundedness, [Bir16a, Lem. 2.22])
*There exists a finite set $I \subset \mathbb{R}$ with the following property. If $X$ is an $\varepsilon$-lc Fano variety of dimension $d-1$, if $r \in \mathbb{R}^{\geq 1}$ and if $D$ is any non-zero integral divisor on $X$ such that $K_X + r \cdot D \equiv 0$, then $r \in I$.* □

We argue by contradiction and assume that there exists a sequence $(X_i)_{i\in\mathbb{N}}$ of $\varepsilon$-lc weak Fanos of dimension $d$ such that the sequence of volumes is strictly increasing, with $\lim \operatorname{vol}(X_i) = \infty$. For simplicity of the argument, assume that all $X_i$ are Fanos rather than weak Fanos, and that they are $\mathbb{Q}$-factorial. For the general case, one needs to consider the maps defined by multiples of $-K_X$ and take small $\mathbb{Q}$-factorialisations.

Choose a rational $\varepsilon'$ in the interval $(0, \varepsilon)$. Using explicit discrepancy computations of boundaries of the form $\frac{1}{N} \cdot B'_i$, for $B'_i \in |-N \cdot K_{X_i}|$ general, [KM98, Cor. 2.32], we find a decreasing sequence $(a_i)_{i\in\mathbb{N}}$ of rationals, with $\lim a_i = 0$, and boundaries $B_i \in \mathbb{Q}\operatorname{Div}(X_i)$ with the following properties.

(6.2.1) For each $i$, the divisor $B_i$ is $\mathbb{Q}$-linearly equivalent to $-a_i \cdot K_{X_i}$.
(6.2.2) The volumes of the $B_i$ are bounded from below, $(2d)^d < \operatorname{vol}(B_i)$.
(6.2.3) The pair $(X_i, B_i)$ has total log discrepancy equal to $\varepsilon'$.

Passing to a subsequence, we may assume that $a_i < 1$ for every $i$. Again, discrepancy computation show that this allows us to find sufficiently general, ample $H_i \in \mathbb{Q}\operatorname{Div}(X_i)$ that are $\mathbb{Q}$-linearly equivalent to $-(1-a_i)\cdot K_{X_i}$ and have the property that $(X, B_i + H_i)$ are still $\varepsilon'$-lc.

Given any index $i$, Item (6.2.3) implies that there exists a prime divisor $D'_i$ on a birational model $X'_i$ that realises the total log discrepancy. For simplicity, consider only the case where one can choose $X_i = X'_i$ for every $i$, and therefore find prime divisors $D_i$ on $X_i$ that appear in $B_i$ with multiplicity $1-\varepsilon'$. Without that simplifying assumption one needs to invoke [BCHM10, Cor. 1.4.3], in order to replace the variety $X_i$ by a model that "extracts" the divisor $D'_i$. In summary, we can write

$$-K_{X_i} \sim_{\mathbb{Q}} \frac{1}{a_i} \cdot B_i = \frac{1-\varepsilon'}{a_i} \cdot D_i + (\text{effective}). \tag{6.2.4}$$

As a next step, recall from Remark 2.15 that the $X_i$ are Mori dream spaces. Given any $i$, we can therefore run the $-D_i$-MMP, which terminates with a Mori fibre space on which the push-forward of $D_i$ is relatively ample. Again, we ignore all technical difficulties and assume that $X_i$ itself is the Mori fibre space, and therefore admits a fibration $X_i \to Z_i$ with relative Picard number $\rho(X_i/Z_i) = 1$ such that $D_i$ is relatively ample. Let $F_i \subseteq X_i$ be a general fibre. Adjunction and standard inequalities for discrepancies imply that $F_i$ is again $\varepsilon$-lc and Fano. The statement about the relative

Picard number implies that any effective divisor on $X_i$ is either trivial or ample on $F_i$. In particular, Equation 6.2.4 implies that $-K_{F_i} \equiv s_i \cdot D_i$, where $s_i \geq \frac{1-\varepsilon'}{a_i}$ goes to infinity. If $\dim F_i = d-1$, or more generally if $\dim F_i < d$ for infinitely many indices $i$, this contradicts Lemma 6.2 and therefore proves Theorem 6.1.

It remains to consider the case where the $Z_i$ are points. Birkar's proof in this case is similar in spirit to the argumentation above, but technically much *more* demanding. He creates a covering family of non-klt centres, uses adjunction on these centres and the assumption that $\varepsilon$-lc Fano varieties of dimension $d-1$ form a bounded family to obtain a contradiction. □

## 7. BOUNDS FOR LC THRESHOLDS

The last of Birkar's core results presented here pertains to log canonical thresholds of anti-canonical systems; this is the main result of Birkar's second paper [Bir16b]. It gives a positive answer to a well-known conjecture of Ambro [Amb16, p. 4419]. With the notation introduced in Section 2.3, the result is formulated as follows.

Theorem 7.1 (Lower bound for lc thresholds, [Bir16b, Thm. 1.4])
*Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, there exists $t \in \mathbb{R}^+$ with the following property. If $(X, B)$ is any projective $\varepsilon$-lc pair of dimension $d$ and if $\Delta := -(K_X + B)$ is nef and big, then* $\operatorname{lct}\big(X,\, B,\, |\Delta|_{\mathbb{R}}\big) \geq t$.

Though this is not exactly obvious, Theorem 7.1 can be derived from boundedness of $\varepsilon$-lc Fanos, Theorem 1.2. One of the core ideas in Birkar's paper [Bir16b] is to go the other way and prove Theorem 7.1 using boundedness, but only for *toric* Fano varieties, where the result has been established by Borisov–Borisov in [BB92].

### 7.1. Idea of application

As pointed out in Section 5.2, bounding lc thresholds from below immediately applies to the boundedness problem. To illustration the application, consider the following corollary, which proves Theorem 1.2 in part.

Corollary 7.2 (Boundedness of $\varepsilon$-lc Fanos). — *Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, the family $\mathcal{X}^{\text{Fano}}_{d,\varepsilon}$ of $\varepsilon$-lc Fanos of dimension $d$ is bounded.*

*Proof.* — We aim to apply Proposition 5.3 to the family $\mathcal{X}^{\text{Fano}}_{d,\varepsilon}$. With Theorem 6.1 ("Bound on volumes") in place, it remains to satisfy Condition (5.3.3) of Proposition 5.3: we need a sequence $(t_\ell)_{\ell \in \mathbb{N}}$ such that the following holds.

> For every $\ell \in \mathbb{N}$, for every $X \in \mathcal{X}^{\text{Fano}}_{d,\varepsilon}$ and every $L \in |-\ell \cdot K_X|$, the pair $(X, t_\ell \cdot L)$ is klt.

But this is not so hard anymore. Let $t \in \mathbb{R}^+$ be the number obtained by applying Theorem 7.1. Given a number $\ell \in \mathbb{N}$, a variety $X \in \mathcal{X}^{\text{Fano}}_{d,\varepsilon}$ and a divisor $L \in |-\ell \cdot K_X|$, observe that $\frac{1}{\ell} \cdot L \in |-K_X|_{\mathbb{R}}$ and recall from Remark 2.10 on page 8 that $(X, \frac{t}{2\ell} \cdot L)$ is klt. We can thus set $t_\ell := \frac{t}{2\ell}$. $\square$

### 7.2. Preparation for the proof of Theorem 7.1: $\mathbb{R}$-linear systems of bounded degrees

To prepare for the proof of Theorem 7.1, we begin with a seemingly weaker result that provides bounds for lc thresholds, but only for $\mathbb{R}$-linear systems of bounded degrees. This result will be used in Section 7.4 to prove Theorem 7.1 in an inductive manner.

Proposition 7.3 (LC thresholds for $\mathbb{R}$-linear systems of bounded degrees, [Bir16b, Thm. 1.6])

*Given $d$, $r \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, there exists $t \in \mathbb{R}^+$ with the following property. If $(X, B)$ is any projective, $\varepsilon$-lc pair of dimension $d$, if $A \in \operatorname{Div}(X)$ is very ample with $A - B$ ample and $[A]^d \leq r$, then $\operatorname{lct}\big(X,\, B,\, |A|_{\mathbb{R}}\big) \geq t$.*

*Remark 7.4.* — The condition on the intersection number, $[A]^d \leq r$ implies that $X$ belongs to a bounded family of varieties. More generally, if we choose $A$ general in its linear system, then $(X, A)$ belongs to a bounded family of pairs.

The proof of Proposition 7.3 is sketched below. It relies on two core ingredients. Because of their independent interest, we formulate them separately.

*Setting 7.5.* — Given $d$, $r \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, we consider projective, $\varepsilon$-lc pairs $(X, B)$ of dimension $d$ where $X$ is $\mathbb{Q}$-factorial, equipped with the following additional data.

(7.5.1) A very ample divisor $A \in \operatorname{Div}(X)$, with $A - B$ ample and $[A]^d \leq r$.
(7.5.2) An effective divisor $L \in \mathbb{R}\operatorname{Div}(X)$, with $A - L$ ample.
(7.5.3) A birational morphism $\nu : Y \to X$ of normal projective varieties, and a prime divisor $T \in \operatorname{Div}(Y)$ whose image is a point $x \in X$.

Lemma 7.6 (Existence of complements, [Bir16b, Prop. 5.9])

*Given $d$, $r \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, assume that Proposition 7.3 holds for varieties of dimension $d-1$. Then, there exist integers $n$, $m \in \mathbb{N}$ and a real number $0 < \varepsilon' < \varepsilon$, with the following property. Whenever we are in Setting 7.5, and whenever there exists a number $t < r$ such that*

(7.6.1) *the pair $(X, B + t \cdot L)$ is $\varepsilon'$-lc, and*
(7.6.2) *the log discrepancy is realised by $T$, that is $a_{\log}(T, X, B + t \cdot L) = \varepsilon'$,*

*Then there exists an effective divisor $\wedge \in \mathbb{Q}\operatorname{Div}(X)$ such that*

(7.6.3) *the divisor $n \cdot \wedge$ is integral,*
(7.6.4) *the tuple $(X, \wedge)$ is lc near $x$, and $T$ is an lc place of $(X, \wedge)$, and*
(7.6.5) *the divisor $m \cdot A - \wedge$ is ample.* $\square$

*Commentary*. — Lemma 7.6 is another existence-and-boundedness result for complements, very much in the spirit of what we have seen in Section 4. The relation to complements is made precise in [Bir16b, Thm. 1.7], which is a core ingredient in Birkar's proof. In fact, after some birational modification of $Y$, Birkar finds a divisor $\wedge_Y \in \operatorname{Div}(Y)$ such that $(Y, \wedge_Y)$ is lc near $T$ and such that $n \cdot (K_Y + \wedge_Y)$ is linearly equivalent to 0, relative to $X$ and for some bounded number $n \in \mathbb{N}$. As Birkar points out in [Bir18, p. 16], one can think of $K_Y + \wedge_Y$ as a local-global type of complement. He then takes $\wedge$ to be the push-forward of $\wedge_Y$ and proves all required properties.

Lemma 7.7 (Bound on multiplicity at an lc place, [Bir16b, Prop. 5.7])

*Given $d$, $r$ and $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, assume that Proposition 7.3 holds for varieties of dimension $\leq d-1$. Then, there exists $q \in \mathbb{R}^+$, with the following property. Whenever we are in Setting 7.5, whenever $a(T, X, B) \leq 1$, and whenever a divisor $\wedge \in \mathbb{Q}\operatorname{Div}(X)$ is given that satisfies the following conditions,*

(7.7.1) *$\wedge$ is effective and $n \cdot \wedge$ is integral,*
(7.7.2) *$A - \wedge$ is ample,*
(7.7.3) *$(X, \wedge)$ is lc near $x$, and $T$ is an lc place of $(X, \wedge)$,*

*then $T$ appears in the divisor $\nu^* L$ with multiplicity $\operatorname{mult}_T \nu^* L \leq q$.* □

*Commentary*. — Lemma 7.7 is perhaps the core of Birkar's paper [Bir16b]. To begin, one needs to realise that the couples $\big(X, \operatorname{supp}(\wedge)\big)$ that appear in Lemma 7.7 come from a bounded family. This allows us to consider common resolution, and eventually to assume from the outset that $(X, \wedge)$ is a log-smooth couple. In particular, $(X, \wedge)$ is toroidal, and $T$ can be obtained by a sequence of blowing ups that are toroidal with respect to $(X, \wedge)$. Given that toroidal blow-ups are rather well understood, Birkar finds that to bound the multiplicity $\operatorname{mult}_T \nu^* L$, it suffices to bound the number of blowups involved.

Bounding the number of blowups is hard, and the next few sentences simplify a very complicated argument to the extreme[(5)]. Birkar establishes a Noether-normalisation theorem, showing that he may replace the couple $(X, \wedge)$, which is log-smooth, by a pair of the form $(\mathbb{P}^d, \text{union of hyperplanes})$, which is toric rather than toroidal. Better still, applying surgery coming from the Minimal Model Programme, he is then able to replace $Y$ by a toric, Fano, $\varepsilon$-lc variety. But the family of such $Y$ is bounded by the classic result of Borisov–Borisov, [BB92], and a bound for the number of blowups follows.

*Sketch of proof for Proposition 7.3*. — The proof of Proposition 7.3 proceeds by induction, so assume that $d$, $r$, and $\varepsilon$ are given and that everything was already shown in lower dimensions. Now, given a $d$-dimensional pair $(X, B)$ and a very ample $A \in \operatorname{Div}(X)$ as in Proposition 7.3, we aim to apply Lemma 7.6 and 7.7. This is, however, not immediately possible because $X$ need not be $\mathbb{Q}$-factorial. We know from minimal model

(5) see [Bir18, p. 16f] and [Xu18, Sect. 10] for a more realistic account of all that is involved.

theory that there exists a small $\mathbb{Q}$-factorialisation, say $X' \to X$, but then we need to compare lc thresholds of $X'$ and $X$, and show that the difference is bounded. To this end, recall from Remark 7.4 that the family of all possible $X$ is bounded, which allows us to construct simultaneous $\mathbb{Q}$-factorialisations in stratified families, and hence gives the desired bound for the differences. Bottom line: we may assume that $X$ is $\mathbb{Q}$-factorial. Let $\varepsilon'$ be the number given by Lemma 7.6.

Next, given any divisor $L \in |A|_{\mathbb{R}}$, look at

$$s := \sup\{s' \in \mathbb{R} \,|\, (X, B + s' \cdot L) \text{ is } \varepsilon'\text{-lc}\}.$$

Following Remark 2.10, we would be done if we could bound $s$ from below, independently of $X$, $B$, $A$ and $L$. To this end, choose a resolution of singularities, $\nu : Y \to X$ and a prime divisor $T \in \operatorname{Div}(Y)$ such that $a_{\log}(T, X, B + s \cdot L) = \varepsilon'$. For simplicity, we will only consider the case where $\nu(T)$ is a point, say $x \in X$ — if $\nu(T)$ is not a point, Birkar cuts down with general hyperplanes from $|A|$, uses inversion of adjunction and invokes the induction hypothesis in order to proceed.

In summary, we are now in a situation where we may apply Lemma 7.6 ("Existence of complements") to find a divisor $\wedge$ and then Lemma 7.7 ("Bound on multiplicity at an lc place") to bound the multiplicity $\operatorname{mult}_T \nu^* L$ from above, independently of $X$, $B$, $A$ and $L$. But then, a look at Definition 2.7 ("log discrepancy") shows that this already gives the desired bound on $s$. □

### 7.3. Preparation for the proof of Theorem 7.1: varieties of Picard-number one

The second main ingredient in the proof of Theorem 7.1 is the following result, which essentially proves Theorem 7.1 in one special case. Its proof, which we do not cover in detail, combines all results discussed in the previous Sections 4–6: boundedness of complements, effective birationality and bounds for volumes.

PROPOSITION 7.8 (Theorem 7.1 in a special case, [Bir16b, Prop 3.1])

*Given $d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$, assume that Proposition 7.3 ("LC thresholds for $\mathbb{R}$-linear systems of bounded degrees") holds in dimension $\leq d$ and that Theorem 1.2 ("Boundedness of $\varepsilon$-lc Fanos") holds in dimension $\leq d - 1$. Then, there exists $v \in \mathbb{R}^+$ such that the following holds. If $X$ is any $\mathbb{Q}$-factorial, $\varepsilon$-lc Fano variety of dimension $d$ of Picard number one, and if $L \in \mathbb{R}\operatorname{Div}(X)$ is effective with $L \sim_{\mathbb{R}} -K_X$, then each coefficient of $L$ is less than or equal to $v$.* □

### 7.4. Sketch of proof of Theorem 7.1

Like other statements, Theorem 7.1 is shown using induction over the dimension. The following key lemma provides the induction step.

LEMMA 7.9 (Implication Proposition 7.3 ⇒ Theorem 7.1, [Bir16b, Lem. 3.2])

*Given $d \in \mathbb{N}$, assume that Proposition 7.3 ("LC thresholds for $\mathbb{R}$-linear systems of bounded degrees") holds in dimension $\leq d$ and that Theorem 1.2 ("Boundedness of*

*$\varepsilon$-lc Fanos") holds in dimension $\le d-1$. Then, Theorem 7.1 ("Lower bound for lc thresholds") holds in dimension $d$.*

*Sketch of proof following* [Bir16b, p. 13f]. — The first steps in the proof are similar to the proof of Proposition 7.3. Choose any number $\varepsilon' \in (0, \varepsilon)$. Given any projective, $d$-dimension, $\varepsilon$-lc pair $(X, B)$ be as in Theorem 7.1 in dimension $d$ and any divisor $L \in |\Delta|_{\mathbb{R}}$, let $s$ be the largest number such that $(X, B + s \cdot L)$ is $\varepsilon'$-lc. We need to show $s$ is bounded from below away from zero. In particular, we may assume that $s < 1$. As in the proof of Proposition 7.3, we may also assume $X$ is $\mathbb{Q}$-factorial. There is a birational modification $\varphi : Y \to X$ and a prime divisor $T \in \operatorname{Div}(Y)$ with log discrepancy

$$a_{\log}(T, X, B + s \cdot L) = \varepsilon'. \tag{7.9.1}$$

Techniques of [BCHM10] ("extracting a divisor") allow us to assume that $\varphi$ is either the identity, or that the $\varphi$-exceptional set equals $T$ precisely. The assumption that $X$ is $\mathbb{Q}$-factorial allows us to pull back divisors. Let

$$B_Y := \varphi^*(K_X + B) - K_Y \quad \text{and} \quad L_Y := \varphi^* L.$$

Using the definition of log discrepancy, Definition 2.7 on page 7, the assumption that $(X, B)$ is $\varepsilon$-lc and Equation (7.9.1) are formulated in terms of divisor multiplicities as

$$\operatorname{mult}_T B_Y \le 1 - \varepsilon \quad \text{and} \quad \operatorname{mult}_T(B_Y + s \cdot L_Y) = 1 - \varepsilon',$$

hence $\operatorname{mult}_T(s \cdot L_Y) \ge \varepsilon - \varepsilon'$.

The pair $(Y, B_Y + s \cdot L_Y)$ is klt and weak log Fano, which implies that $Y$ is Fano type. Recalling from Remark 2.15 on page 9 that $Y$ is thus a Mori dream space, we may run a $(-T)$-Minimal Model Programme and obtain rational maps,

$$Y \xrightarrow{\alpha,\ \text{extr. contractions and flips}} Y' \xrightarrow{\beta,\ \text{Mori fibre space}} Z',$$

where $-T$ is ample when restricted to general fibres of $\beta$. We write $B_{Y'} := \alpha_* B_Y$ and $L_{Y'} := \alpha_* L_Y$ and note that

$$-(K_{Y'} + B_{Y'} + s \cdot L_{Y'}) \overset{\text{def. of } L}{\sim_{\mathbb{R}}} (1-s) L_{Y'} \overset{s<1}{\ge} 0.$$

Moreover, an explicit discrepancy computation along the lines of [KM98, Cor. 2.32] shows that $(Y', B_{Y'} + s \cdot L_{Y'})$ is $\varepsilon'$-lc, because $(Y, B_Y + s \cdot L_Y)$ is $\varepsilon'$-lc and because $-(K_Y + B_Y + s \cdot L_Y)$ is semiample. There are two cases now.

If $\dim Z' > 0$, then restricting to a general fibre of $Y' \to Z'$ and applying Proposition 7.3 ("LC thresholds for $\mathbb{R}$-linear systems of bounded degrees") in lower dimension(6) shows that the coefficients of those components of $(1-s) \cdot L_{Y'}$ that dominate $Z'$ components of are bounded from above. In particular, $\operatorname{mult}_{T'}(1-s) \cdot L_{Y'}$ is bounded from above. Thus from the inequality

$$\operatorname{mult}_{T'}(1-s) \cdot L_{Y'} \ge \frac{(1-s) \cdot (\varepsilon - \varepsilon')}{s},$$

(6) or applying Theorem 1.2 ("Boundedness of $\varepsilon$-lc Fanos")

we deduce that $s$ is bounded from below away from zero.

If $Z'$ is a point, then $Y'$ is a Fano variety with Picard number one. Now

$$-K_{Y'} \sim_{\mathbb{R}} (1-s) \cdot L_{Y'} + B_{Y'} + s \cdot L_{Y'} \geq (1-s) \cdot L_{Y'},$$

so by Proposition 7.8, $\mathrm{mult}_{T'}(1-s) \cdot L_{Y'}$ is bounded from above which again gives a lower bound for $s$ as before. □

## 8. APPLICATION TO THE JORDAN PROPERTY

We explain in this section how the boundedness result for Fano varieties applies to the study of birational automorphism groups, and how it can be used to prove the Jordan property. Several of the core ideas presented here go back to work of Serre, who solved the two dimensional case, [Ser09, Thm. 5.3] but see also [Ser10, Thm. 3.1]. If one is only interested in the three-dimensional case, where birational geometry is particularly well-understood, most arguments presented here can be simplified.

### 8.1. Existence of subgroups with fixed points

If $X$ is any rationally connected variety, Theorem 1.4 ("Jordan property of Cremona groups") asks for the existence of finite Abelian groups in the Cremona groups $\mathrm{Bir}(X)$. As we will see in the proof, this is almost equivalent to asking for finite groups of automorphisms that admit fixed points, and boundedness of Fanos is the key tool used to find such groups. The following lemma is the simplest result in this direction. Here, boundedness enters in a particularly transparent way.

LEMMA 8.1 (Fixed points on Fano varieties, [PS16, Lem. 4.6])

*Given $d \in \mathbb{N}$, there exists a number $j_d^{\mathrm{Fano}} \in \mathbb{N}$ such that for any $d$-dimensional Fano variety $X$ with canonical singularities and any finite subgroup $G \subseteq \mathrm{Aut}(X)$, there exists a subgroup $F \subseteq G$ of index $|G : F| \leq j_d^{\mathrm{Fano}}$ acting on $X$ with a fixed point.*

*Remark 8.2.* — To keep notation simple, Lemma 8.1 is formulated for Fanos with canonical singularities, which is the relevant case for our application. In fact, it suffices to consider Fanos that are $\varepsilon$-lc.

*Proof of Lemma 8.1.* — As before, write $\mathcal{X}_{d,0}^{\mathrm{Fano}}$ for the $d$-dimensional Fano variety $X$ with canonical singularities. It follows from boundedness, Theorem 1.2 or Corollary 7.2, that there exist numbers $m$, $v \in \mathbb{N}$ such that the following holds for every $X \in \mathcal{X}_{d,0}^{\mathrm{Fano}}$.

(8.2.1) The divisor $-m \cdot K_X$ is Cartier and very ample.

(8.2.2) The self-intersection number of $-m \cdot K_X$ is bounded by $v$. More precisely,

$$-[m \cdot K_X]^d \leq v.$$

Given $X$, observe that the associated line bundles $\mathscr{O}_X(-m \cdot K_X)$ are $\mathrm{Aut}(X)$-linearised. Accordingly, there exists a number $N \in \mathbb{N}$, and for every $X \in \mathcal{X}^{\mathrm{Fano}}_{d,0}$ an $\mathrm{Aut}(X)$-equivariant embedding $X \hookrightarrow \mathbb{P}^N$. Let $j^{\mathrm{Jordan}}_{N+1}$ be the number obtained by applying the classical result of Jordan, Theorem 1.6, to $\mathrm{GL}_{N+1}(\mathbb{C})$, and set $j^{\mathrm{Fano}}_d := j^{\mathrm{Jordan}}_{N+1} \cdot v$.

Now, given any $X \in \mathcal{X}^{\mathrm{Fano}}_{d,0}$ and any finite subgroup $G \subseteq \mathrm{Aut}(X)$, the $G$ action extends to $\mathbb{P}^N$. The action is thus induced by a representation of a finite linear group $\Gamma$, say

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d] & \mathrm{GL}_{N+1}(\mathbb{C}) \arrow[d] \\ G \arrow[r, hook] & \mathbb{P}\,\mathrm{GL}_{N+1}(\mathbb{C}). \end{tikzcd}$$

By Theorem 1.6, the classic result of Jordan, we find a finite Abelian subgroup $\Phi \subseteq \Gamma$ of index $|\Phi : \Gamma| \leq j^{\mathrm{Jordan}}_{N+1}$. Since $\Phi$ is Abelian, the $\Phi$-representation space $\mathbb{C}^{N+1}$ is a direct sum of one-dimensional representations. Equivalently, we find $N+1$ linearly independent, $\Phi$-invariant, linear hyperplanes $H_i \subset \mathbb{P}^{N+1}$. The intersection of suitably chosen $H_i$ with $X$ is then a finite, $\Phi$-invariant subset $\{x_1, \ldots, x_r\} \subset X$, of cardinality $r \leq v$. The stabiliser of $x_1 \in X$ is a subgroup $\Phi_{x_1} \subset \Phi$ of index $|\Phi : \Phi_{x_1}| \leq v$. Taking $F$ as the image of $\Phi_{x_1} \to G$, we obtain the claim. □

As a next step, we aim to generalise the results of Lemma 8.1 to varieties that are rationally connected, but not necessarily Fano. The following result makes this possible.

Lemma 8.3 (Rationally connected subvarieties on different models, [PS16, Lem. 3.9])
*Let $X$ be a projective variety with an action of a finite group $G$. Suppose that $X$ is klt, with $G\mathbb{Q}$-factorial singularities and let $f : X \dashrightarrow Y$ be a birational map obtained by running a $G$-Minimal Model Programs. Suppose that there exists a subgroup $F \subset G$ and an $F$-invariant, rationally connected subvariety $T \subsetneq Y$. Then, there exists an $F$-invariant rationally connected subvariety $Z \subsetneq X$.* □

Since we are mainly interested to see how boundedness applies to birational transformation groups, we will not explain the proof of Lemma 8.3 in detail. Instead, we merely list a few of the core ingredients, which all come from minimal model theory and birational geometry.

– Hacon–McKernan's solution [HM07] to Shokurov's "rational connectedness conjecture", which guarantees in essence that the fibres of all morphisms appearing in the MMP are rationally chain connected.
– A fundamental result of Graber–Harris–Starr, [GHS03], which implies that if $f : X \to Y$ is any dominant morphism of proper varieties, where both the target $Y$ and a general fibre is rationally connected, then $X$ is also rationally connected.
– Log-canonical centre techniques, in particular a relative version of Kawamata's subadjunction formula, [PS16, Lem. 2.5]. These results identify general fibres of minimal log-canonical centres under contraction morphisms as rationally connected varieties of Fano type.

PROPOSITION 8.4 (Fixed points on rationally connected varieties, [PS16, Lem. 4.7])
*Given $d \in \mathbb{N}$, there exists a number $j^{rc}_d \in \mathbb{N}$ such that for any d-dimensional, rationally connected projective variety $X$ and any finite subgroup $G \subseteq \operatorname{Aut}(X)$, there exists a subgroup $F \subseteq G$ of index $|G : F| \leq j^{rc}_d$ acting on $X$ with a fixed point.*

*Sketch of proof.* — We argue by induction on the dimension. Since the case $d = 1$ is trivial, assume that $d > 1$ is given, and that numbers $j^{rc}_1, \dots, j^{rc}_{d-1}$ have been found. Set

$$j_d := \max\{j^{rc}_1, \dots, j^{rc}_{d-1}, j^{\text{Fano}}_d\} \quad \text{and} \quad j^{rc}_d := (j_d)^2.$$

Assume that a $d$-dimensional, rationally connected projective variety $X$ and a finite subgroup $G \subseteq \operatorname{Aut}(X)$ are given. By induction hypothesis, it suffices to find a subgroup $G' \subseteq G$ of index $|G : G'| \leq j_d$ and a $G'$-invariant, rationally connected, proper subvariety $X' \subsetneq X$.

If $\widetilde{X} \to X$ is the canonical resolution of singularities, as in [BM97], then $\widetilde{X}$ is likewise rationally connected, $G$ acts on $\widetilde{X}$ and the resolution morphism is equivariant. Since images of rationally connected, invariant subvarieties are rationally connected and invariant, we may assume from the outset that $X$ is smooth. But then we can run a $G$-equivariant Minimal Model Programme[(7)] terminating with a $G$-Mori fibre space,

$$X \xrightarrow{G\text{-equivariant MMP}} X' \xrightarrow{G\text{-Mori fibre space}} Y.$$

In the situation at hand, Lemma 8.3 claims that to find proper, invariant, rationally connected varieties on $X$, it is equivalent to find them on $X'$. The fibre structure, however, makes that feasible.

Indeed, if the base $Y$ of the fibration happens to be a point, then $X'$ is Fano with terminal singularities, and Lemma 8.1 applies. Otherwise, let $G_Y$ be the image of $G$ in $\operatorname{Aut}(Y)$, let $G_{X'/Y} \subseteq G$ be the ineffectivity of the $G$-action on $Y$, and consider the exact sequence

$$1 \to G_{X'/Y} \to G \to G_Y \to 1.$$

As the image of the rationally connected variety $X'$, the base $Y$ is itself rationally connected. By induction hypothesis, using that $\dim Y < \dim X$, there exists a subgroup $F'_Y \subseteq G_Y$ of index $|G_Y : F'_Y| < j_d$ that acts on $Y$ with a fixed point, say $y \in Y$. Let $G' \subset G$ be the preimage of $G'_Y$. The fibre $X_y$ is then invariant with respect to the action of $G'$ and rationally chain connected by [HM07, Cor. 1.3]. Better still, Prokhorov and Shramov show that it contains a rationally connected, $G'$-invariant subvariety. The induction applies. □

(7) The existence of an MMP terminating with a fibre space is [BCHM10, Cor. 1.3.3], which we have quoted before. The fact that the MMP can be chosen in an equivariant manner is not explicitly stated there, but follows without much pain.

### 8.2. Proof of Theorem 1.4 ("Jordan property of Cremona groups")

Given a number $d \in \mathbb{N}$, we claim that the number $j := j_d^{rc} \cdot j_d^{\mathrm{Jordan}}$ will work for us, where $j_d^{rc}$ is the number found in Proposition 8.4, and $j_d^{\mathrm{Jordan}}$ comes from Jordan's Theorem 1.6. To this end, let $X$ be any rationally connected variety of dimension $d$, and let $G \subseteq \mathrm{Bir}(X)$ be any finite group. Blowing up the indeterminacy loci of the birational transformations $g \in G$ in an appropriate manner, we find a birational, $G$-equivariant morphism $\widetilde{X} \to X$ where the action of $G$ in $\widetilde{X}$ is regular rather than merely birational, see [Sum74, Thm. 3]. Combining with the canonical resolution of singularities, we may assume that $\widetilde{X}$ is smooth. Proposition 8.4 will then guarantee the existence of a subgroup $G' \subseteq G$ of index $|G : G'| \leq j_d^{rc}$ acting on $\widetilde{X}$ with a fixed point $\widetilde{x}$. Standard arguments ("linearisation at a fixed point") that go back to Cartan, cf. [FZ05, Lem. 2.7(b)] or [HO84, p. 11ff], show that the induced action of $G'$ on the Zariski tangent space $T_{\widetilde{x}}(\widetilde{X})$ is faithful, so that Jordan's Theorem 1.6 applies. □

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