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HIGHER RANK TEICHMÜLLER THEORIES

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INTRODUCTION

Let Γ_g be the fundamental group of a compact surface S_g with negative Euler characteristic, and let G denote $\mathrm{PSL}(2, \mathbf{R})$, the group of isometries of the hyperbolic plane $\mathbf{H}^2$. Goldman observed that the Teichmüller space, the parameter space of marked hyperbolic structures on S_g, can be identified with a connected component of the character variety $\mathrm{Hom}(\Gamma_g, G)/\!\!/G$, which can be selected by means of a characteristic invariant. Thanks to the work of Labourie, Burger–Iozzi–Wienhard, Fock–Goncharov and Guichard–Wienhard we now know that, surprisingly, this is a much more general phenomenon: there are also many higher rank semisimple Lie groups G admitting components of the character variety consisting only of injective homomorphisms with discrete image, the so-called *higher rank Teichmüller theories*. The richness of these theories is partially due to the fact that, as for the Teichmüller space, truly different techniques can be used to study them: bounded cohomology, Higgs bundles, positivity, harmonic maps, incidence structures, geodesic currents, real algebraic geometry, dynamics are just some of those.

In this survey, after introducing the two known families of higher rank Teichmüller theories, the Hitchin components and the maximal representations, we will describe a conjectural unifying framework, Θ-positive representations. This theory, due to Guichard–Wienhard, encompasses both families of higher rank Teichmüller theories as well as, potentially, new families associated to orthogonal groups. In Section 3, after reinterpreting Higher Teichmüller theories as moduli spaces of locally symmetric spaces, we will discuss several geometric properties of such locally symmetric spaces, highlighting analogies and differences with geometric properties of hyperbolic surfaces, points in the Teichmüller spaces. We will be particularly concerned with the (vector valued) length functions associated to these locally symmetric spaces, and with various techniques to study them, based on dynamics, as well as incidence geometry and positivity. After a short digression, in Section 4, on harmonic maps and minimal surfaces, which provide a more analytic tool to study higher rank Teichmüller theories, we will focus, in the last section of the survey, on the interplay with other geometric structures, particularly in rank 2.

This short survey is not intended to be exhaustive, but it is rather a concise description of a few of the many ideas and tools that are being developed in the study of character varieties. In particular, for lack of space, we decided not to discuss the related theory of Anosov representations nor to detail the Higgs bundles perspective on character varieties and higher rank Teichmüller theories. We will instead discuss some of the applications of the theory of Higgs bundles, emphasizing results which can be formulated purely in terms of synthetic geometry, despite the only available proofs make heavy use of the more analytic approach. We refer the reader to the surveys [Ale18, Gar19] and references therein for an introduction to Higgs bundles and their use in the study of character varieties, to the survey [Wie18] for a discussion of other aspects of higher rank Teichmüller theories, and to the surveys [Gui17, Kas18] for an introduction to Anosov representations and their link with geometric structures.

1. TEICHMÜLLER THEORY

In this section we recall some basic facts about Teichmüller theory that will play an important role in the higher rank generalizations that we will discuss in the rest of the survey. We refer the reader to [FM12, Part 2] for an introduction to these themes close to the viewpoint we will follow here.

Let S_g be a closed oriented surface of genus $g \geq 2$. We will define the *Teichmüller space* $\mathcal{T}(S_g)$ as the space of homotopy classes of marked hyperbolic structures on S_g.(1) The Teichmüller space is isomorphic to $\mathbf{R}^{6g-6}$, as can be seen using Fenchel–Nielsen coordinates: the choice of a maximal collection $\{c_1, \dots, c_{3g-3}\}$ of pairwise disjoint simple closed curves decomposes the surface S_g as a union of pairs of pants $\{P_1, \dots, P_{2g-2}\}$; the parametrization of $\mathcal{T}(S_g)$ can then be obtained recording the $3g-3$ lengths of the curves c_i and how much twist is involved in the glueings; indeed any three holed sphere (pair of pants) admits a unique hyperbolic structure for each choice of boundary lengths.

Whenever we fix a hyperbolic metric h on S_g, we can identify the metric universal covering $(\widetilde{S}_g, \widetilde{h})$ with the hyperbolic plane $\mathbf{H}^2$; the identification is natural up to post-composition with an element in $\mathrm{PSL}_2(\mathbf{R})$, the group of orientation-preserving isometries of $\mathbf{H}^2$. Throughout the survey we will denote by Γ_g the fundamental group of the surface S_g. The action of Γ_g on $\widetilde{S}_g$ as deck transformations induces, via the identification $\widetilde{S}_g \cong \mathbf{H}^2$, a homomorphism $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbf{R})$, which is then well-defined up to conjugation in $\mathrm{PSL}(2, \mathbf{R})$. This homomorphism is called the *holonomy* of the hyperbolic structure (S_g, h).

We will denote by $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R})) /\!\!/ \mathrm{PSL}_2(\mathbf{R})$ the *character variety*, namely the largest Hausdorff quotient of the set of homomorphisms $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbf{R})$ for the

1. This is historically inaccurate, as the Teichmüller space is the space of marked conformal structures on S_g, while the space of marked hyperbolic structures should be referred to as *Fricke space*. However it is a consequence of the uniformization theorem that these two objects can be identified.

equivalence relation that identifies two homomorphisms ρ, η if there exists $g \in \mathrm{PSL}(2, \mathbf{R})$ such that for every $\gamma \in \Gamma_g$, $\rho(\gamma) = g\eta(\gamma)g^{-1}$. The choice of a finite generating set S of Γ_g allows to realise $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))$ as a subset of $\mathrm{PSL}_2(\mathbf{R})^{|S|}$ defined by polynomial equations (induced by the relations of the group Γ_g); this also induces a natural semi-algebraic structure on the character variety [Bru88].

It is a basic fact in covering theory that the homomorphisms ρ arising as holonomies of hyperbolizations are injective and have discrete image. In his thesis Goldman showed that this procedure actually gives an identification of the Teichmüller space $\mathcal{T}(S_g)$ with a connected component of the character variety $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))/\!\!/\mathrm{PSL}_2(\mathbf{R})$, which can be selected by means of a cohomological invariant, the Euler class. [(2)]

Theorem 1.1 (Goldman [Gol80]). — *The Euler class* $\mathrm{eu}(\rho)$ *distinguishes connected components in* $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbf{R}))/\!\!/\mathrm{PSL}_2(\mathbf{R})$ *and has values in* $\mathbf{Z} \cap [\chi(S_g), -\chi(S_g)]$. *The representations for which* $|\mathrm{eu}(\rho)|$ *is maximal correspond to holonomies of hyperbolic structures on* S_g *(resp. hyperbolic structures on* S_g *endowed with the opposite orientation).*

It could be natural to think that there are connected components of the $\mathrm{PSL}(2, \mathbf{R})$-character variety only consisting of injective homomorphisms with discrete image because the cohomological dimension of the group Γ_g equals the dimension of $\mathbf{H}^2$ and thus Γ_g can act properly discontinuously and co-compactly on $\mathbf{H}^2$. We will discuss in the rest of the survey that the existence of such components is, instead, a much more general phenomenon: there are various classes of semisimple Lie groups G for which $\mathrm{Hom}(\Gamma_g, G)/\!\!/G$ has connected components only consisting of injective homomorphisms with discrete image, the so-called *Higher Teichmüller theories*.

A lot of the richness of Teichmüller theory can be tracked back to the local isogenies between semisimple Lie groups in low ranks: $\mathrm{PSL}(2, \mathbf{R})$ is isomorphic to $\mathrm{PSp}(2, \mathbf{R})$, $\mathrm{PU}(1, 1)$ and $\mathrm{PO}(2, 1)$. In turn these correspond to different models for the hyperbolic plane (respectively, the upper-half plane $\mathbf{H}^2 \subset \mathbf{C}$, the Poincaré disk $\mathbf{D} \subset \mathbf{CP}^1$, and the Klein model $\mathbf{K} \subset \mathbf{RP}^2$) and therefore different perspectives on the same theory. We won't have this at our disposal when dealing with a general Lie group G, but we will discuss in Section 5 how the interplay between different geometric structures associated to G can give new insight on representations in Higher Teichmüller theories.

We conclude our very short account of Teichmüller theory by discussing an important property of hyperbolizations, which will have an avatar of fundamental importance in higher rank Teichmüller theory: the existence of boundary maps. Recall that the hyperbolic plane $\mathbf{H}^2$ has a boundary $\partial_\infty \mathbf{H}^2$ isomorphic to the circle $\mathbf{S}^1$ and consisting of equivalence classes of asymptotic rays. Given any two hyperbolic structures (S_g, h_1), (S_g, h_2) on the surface S_g with holonomies ρ_i, we obtain, via the identifications $(\widetilde{S}_g, \widetilde{h}_i) \cong \mathbf{H}^2$, a continuous (ρ_1, ρ_2)-equivariant map $f_{\rho_1, \rho_2} : \mathbf{H}^2 \to \mathbf{H}^2$. This extends to a monotone, Hölder continuous map $\xi_{\rho_1, \rho_2} : \partial_\infty \mathbf{H}^2 \to \partial_\infty \mathbf{H}^2$. Here monotonicity is defined with

2. We will not need the definition of the Euler class in the rest of the text. The interested reader can read more about it for example in [BIW14].

respect to the cyclic orientation of the circle: the map ξ_{ρ_1,ρ_2} is monotone if for every positively oriented triple (x, y, z) the image $(\xi_{\rho_1,\rho_2}(x), \xi_{\rho_1,\rho_2}(y), \xi_{\rho_1,\rho_2}(z))$ is positively oriented. We fix for simplicity (3) an auxiliary hyperbolic structure on the surface S_g with holonomy ρ, and denote by $\partial_\infty\Gamma_g$ the boundary $\partial_\infty\mathbf{H}^2$ together with the action of Γ_g induced by ρ. The discussion above shows that the Hölder structure of $\partial_\infty\Gamma_g$, as well as its cyclic order, is intrinsic and doesn't depend on the choice of ρ. It is then possible to characterise holonomies of hyperbolization using boundary maps: a representation $\eta : \Gamma_g \to \mathrm{PSL}(2, \mathbf{R})$ is the holonomy of a hyperbolization if and only if there exists a monotone, Hölder continuous map $\xi_\eta : \partial_\infty\Gamma_g \to \partial_\infty\mathbf{H}^2$.

2. HIGHER RANK TEICHMÜLLER THEORIES

Let us now consider a connected, adjoint, semisimple Lie group G of non-compact type and higher rank. Natural examples that will play a role in the text are $\mathrm{PSL}(n, \mathbf{R})$, the projective classes of matrices of determinant one, $\mathrm{PSp}(2n, \mathbf{R})$, the projective classes of matrices of determinant one preserving a symplectic form on $\mathbf{R}^{2n}$ or $\mathrm{PO}_0(2, n)$, the projectivization of the connected component of the identity in the group preserving a symmetric bilinear form on $\mathbf{R}^{n+2}$ of signature $(2, n)$. In this survey we will mostly regard G as the identity component of the group $\mathrm{Isom}(\mathcal{X})$, where $\mathcal{X} = G/K$ is the Riemannian symmetric space associated to G, a non-positively curved Riemannian manifold in which the geodesic reflections about any point are induced by isometries.

Definition 2.1. — *A* higher rank Teichmüller theory *is a connected component of the character variety* $\mathrm{Hom}(\Gamma_g, G)/\!\!/G$ *only consisting of injective homomorphisms with discrete image.*

For most of the survey we will think of such higher rank Teichmüller theories as parametrizing special classes of locally symmetric spaces $\rho(\Gamma_g)\backslash\mathcal{X}$ covered by the Riemannian symmetric space $\mathcal{X}$, and whose fundamental group is Γ_g. Observe that, being a connected component of a character variety in a reductive algebraic group, any higher rank Teichmüller theory has a natural structure of a real semi-algebraic variety, and is thus amenable to the tools of real semi-algebraic geometry [Bru88, Ale08, FG06].

2.1. Hitchin components

Let G be a real split simple Lie group, such as $\mathrm{PSL}(n, \mathbf{R})$ or $\mathrm{PSp}(2n, \mathbf{R})$. By the work of Kostant [Kos59] there exists a principal homomorphism $\tau : \mathrm{PSL}(2, \mathbf{R}) \to G$: the unique homomorphism for which the image of a diagonalizable element (and thus any diagonalizable element) is diagonalizable with distinct eigenvalues. If $G = \mathrm{PSL}(n, \mathbf{R})$, the principal homomorphism is the irreducible representation $\tau : \mathrm{PSL}(2, \mathbf{R}) \to \mathrm{PSL}(n, \mathbf{R})$

3. With basic tools of geometric group theory one can give an intrinsic definition of the boundary $\partial_\infty\Gamma_g$, but this won't be necessary for our purposes.

induced by the natural action of $\mathrm{PSL}(2,\mathbf{R})$ on the homogeneous polynomials of degree $n-1$.

In [Hit92] Hitchin initiated the study of the connected component in $\mathrm{Hom}(\Gamma_g, G)/\!\!/G$ of the composition $\tau\circ\rho$ where $\rho:\Gamma_g\to\mathrm{PSL}(2,\mathbf{R})$ is the holonomy of a hyperbolization:

Definition 2.2. — *Let G be a real split simple Lie group, $\tau:\mathrm{PSL}(2,\mathbf{R})\to G$ the principal homomorphism, $\rho:\Gamma_g\to\mathrm{PSL}(2,\mathbf{R})$ the holonomy of a hyperbolization. The* Hitchin component $\mathrm{Hit}(\Gamma_g, G)$ *is the connected component of* $[\tau\circ\rho]$ *in* $\mathrm{Hom}(\Gamma_g, G)/\!\!/G$.

Using analytic techniques, and in particular the theory of Higgs bundles developed by Hitchin [Hit87], Simpson [Sim88, Sim92], Corlette [Cor88] and Donaldson [Don87], Hitchin was able to show that, as in the case of Teichmüller space, the Hitchin component $\mathrm{Hit}(\Gamma_g, G)$ is homeomorphic to the Euclidean space of dimension $(2g-2)\dim G$.

The geometric relevance of representations in $\mathrm{Hit}(\Gamma_g,\mathrm{PSL}(n,\mathbf{R}))$ was singled out by Labourie using dynamical techniques: in [Lab06] Labourie introduced the notion of Anosov representation, and proved that representations in the Hitchin component are injective, have discrete image, and are purely loxodromic, which means that for every element $\gamma\in\Gamma_g$, the image $\rho(\gamma)$ is diagonalizable with distinct real eigenvalues. In particular Hitchin components form examples of higher rank Teichmüller theories, according to Definition 2.1.

An independent approach to the study of Hitchin components was developed by Fock and Goncharov [FG06], based on Lusztig's generalization [Lus94] of the notion of totally positive matrices, namely matrices whose minors are all positive. Lusztig associated to every split semisimple real Lie group G a positive submonoid $G_{>0}$ with remarkable algebraic properties. Let $\mathcal{F}=G/B$ denote the full flag variety associated to G, which, in the case of $G=\mathrm{PSL}(n,\mathbf{R})$, is nothing but the collection of full flags $\{0\}=V^{(0)}\subsetneq V^{(1)}\subsetneq\ldots\subsetneq V^{(n)}=\mathbf{R}^n$. Fock and Goncharov [FG06] used the tools provided by Lusztig's theory of positivity to define the notion of positivity for a k-tuple $(F_1,\ldots,F_k)$ of flags in $\mathcal{F}$. This began the study of the space of *positive decorated representations*: representations $\rho:\Gamma_g\to G$ admitting a positive decoration, a map from the cyclically ordered set $X\subset\partial\Gamma_g$ of fixed points of hyperbolic elements to $\mathcal{F}$ for which the image of any cyclically ordered k-tuple is a positive k-tuple of flags. [4]

One of the crucial differences between higher rank symmetric spaces and their rank one analogues, such as the hyperbolic plane, is that the visual boundary $\partial_\infty\mathcal{X}$ of the symmetric space is not anymore a homogeneous G-space, but stratifies in orbits isomorphic to partial flag varieties: compact homogeneous G-spaces G/P, determined by the choice of a parabolic subgroup P. When considering boundary maps, it is thus natural to consider, instead of maps $\xi:\partial\Gamma_g\to\partial_\infty\mathcal{X}$, maps of the form $\xi_P:\partial\Gamma_g\to G/P$ for a suitable choice of a parabolic subgroup P.

4. The work of Fock and Goncharov mostly deals with representations of surfaces with punctures, in which a decoration is only required above the punctures.

It was proven by Labourie and Guichard, that Hitchin representations in $\mathrm{PSL}(n,\mathbf{R})$ can be characterised by the properties of the boundary map they admit with value in the projective space $\mathbf{RP}^{n-1}$. We say that the map $\xi : \partial\Gamma_g \to \mathbf{RP}^{n-1}$ is *hyperconvex* if for every pairwise distinct points $x_1, \dots, x_n \in \partial\Gamma_g$ the sum $\bigoplus_{k=1}^n \xi(x_k)$ is direct.

THEOREM 2.1 (Labourie [Lab06], Guichard [Gui08]). — *Let $\rho : \Gamma_g \to \mathrm{PSL}(d,\mathbf{R})$ be a homomorphism. Then $[\rho]$ belongs to the Hitchin component $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$ if and only if there exists a continuous ρ-equivariant hyperconvex map $\xi : \partial\Gamma_g \to \mathbf{RP}^{n-1}$.*

More precisely Labourie proved that Hitchin representations admit equivariant hyperconvex boundary maps, while Guichard proved that this property is enough to characterise such representations. Hitchin representations also admit continuous boundary maps with values in the full flag manifold $\mathcal{F}$ extending the decoration described above.

2.2. Maximal representations

Let now G be an Hermitian Lie group, such as $\mathrm{PSp}(2n,\mathbf{R})$ or $\mathrm{PO}_0(2,n)$. By definition the symmetric space $\mathcal{X}$ admits a G-invariant complex structure, and is thus a Kähler manifold with Kähler form ω. It is possible to define the volume of a representation ρ by setting

$$T(\rho) = \frac{1}{2\pi}\int_{S_g} \pi_* f^*\omega$$

where $f : \widetilde{S}_g \to \mathcal{X}$ is any smooth ρ-equivariant map and $\pi : \widetilde{S}_g \to S_g$ is the universal covering. The characteristic number $T(\rho)$ is the *Toledo invariant* of the representation ρ, it generalises the Euler number $\mathrm{eu}(\rho)$ appearing in Theorem 1.1, and, as the Euler number, it satisfies a *Milnor–Wood inequality*:

$$|T(\rho)| \le (2g-2)\mathrm{rk}_{\mathbf{R}}(G)$$

where the real rank $\mathrm{rk}_{\mathbf{R}}(G)$ is the maximal dimension of a flat subspace of $\mathcal{X}$ [BIW14, Proposition 3.1].

DEFINITION 2.3. — *Let G be a Hermitian Lie group. A representation $\rho : \Gamma_g \to G$ is* maximal *if its Toledo invariant satisfies the equality in the Milnor–Wood inequality. We will denote by $\mathrm{Max}(\Gamma_g, G) \subset \mathrm{Hom}(\Gamma_g, G)/\!\!/G$ the set of maximal representations.*

The Toledo invariant for representations $\rho : \Gamma_g \to \mathrm{PU}(1,n)$ was first introduced by Toledo [Tol89] who used it to prove rigidity for Fuchsian subgroups acting on complex hyperbolic spaces. Burger, Iozzi and Wienhard [BILW05, BIW10] initiated the study of the Toledo invariant for general Hermitian Lie groups in the framework of bounded cohomology. With this tool they proved that maximal representations are other instances of higher rank Teichmüller theories: they form unions of connected components of the character variety consisting of classes of injective homomorphisms with discrete image. Furthermore an analogue of Theorem 2.1 holds for maximal representations as well. In this case the suitable parabolic to consider is the stabiliser Q of a point in the Shilov boundary Š of the Hermitian symmetric space (this is the set of Lagrangians in the case of $G = \mathrm{PSp}(2n, \mathbf{R})$ and the set of isotropic lines in the case of $G = \mathrm{PO}_0(2, n)$). The Maslov cocycle induces a partial cyclic order on Š, and maximal representations can be characterised as those representations admitting a *monotone* equivariant boundary map, namely a map $\xi : \partial\Gamma_g \to$ Š such that for every positively oriented triple $(x, y, z) \in \partial\Gamma_g^3$ the image $(\xi(x), \xi(y), \xi(z))$ is Maslov-positively oriented.

Using the theory of Higgs bundles together with ideas from [Hit92], Bradlow, García-Prada and Gothen [BGPG06] managed to count the connected components of $\mathrm{Max}(\Gamma_g, G)$ and showed that the components can be selected with the aid of secondary characteristic invariants; in the case of $G = \mathrm{PSp}(2n, \mathbf{R})$, Guichard–Wienhard gave another interpretation of the invariants distinguishing components of maximal representations based on properties of the associated boundary map [GW10]. Apart from $2g - 3$ exceptional components in $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$, that are often referred to as the *Gothen components* and consist entirely of Zariski-dense representations, every maximal component admits a Fuchsian locus, consisting of representations that preserve a totally geodesic copy of $\mathbf{H}^2$ in $\mathcal{X}$ whose quotient modulo the representation is a point in the Teichmüller space $\mathcal{T}(S_g)$. Model representations in the Gothen components have been constructed by Guichard–Wienhard as amalgams of representations [GW10] and by Kydonakis by establishing a gluing construction for Higgs bundles over a connected sum of Riemann surfaces [Kyd18].

The only family of split simple Lie groups of Hermitian type is given by $\mathrm{PSp}(2n, \mathbf{R})$. In this case Hitchin representations are maximal, but the set of maximal representations includes more connected components of the character variety [BILW05].

2.3. Θ-positive representations

A common framework explaining the various higher rank Teichmüller theories is now emerging thanks to the work of Guichard–Wienhard [GW18] and work in progress of Guichard–Labourie–Wienhard. Let G be a semisimple Lie group with finite center and P_Θ a self-opposite parabolic subgroup corresponding to a subset Θ of the simple roots. Given $E \in G/P_\Theta$ denote by $(G/P_\Theta)^E$ the set of points F in G/P_Θ that are transverse [(5)] to E. Guichard–Wienhard say that the group G admits a Θ-positive structure if there are two transverse points $E, F \in G/P_\Theta$ such that a connected component of $(G/P_\Theta)^E \cap (G/P_\Theta)^F$ has the structure of a semigroup. In this case one can talk about Θ-positive triples, and define

Definition 2.4. — *Let G be a semisimple Lie group with a Θ-positive structure. A representation $\rho : \Gamma_g \to G$ is* Θ-positive *if there exists a ρ-equivariant boundary map $\xi : \partial\Gamma_g \to G/P_\Theta$ sending positive triples to Θ-positive triples.*

5. This means that the pair (E, F) belongs to the unique open G-orbit in $G/P_\Theta \times G/P_\Theta$. This notion agrees with the standard notion of transversality when G/P_Θ corresponds to a partial flag manifold.

Guichard–Labourie–Wienhard conjecture that Θ-positive representations also form higher rank Teichmüller spaces, namely they form connected components of the character variety consisting of injective representations with discrete image.

Using a more algebraic characterisation of Θ-positivity, inspired by the work of Lustzig, Guichard and Wienhard classify the groups G admitting a Θ-positive structure. They show that split real Lie groups admit a Θ-positive structure induced by Lustzig positivity, and for such structure Θ-positive representations are Hitchin representations, similarly Hermitian Lie groups have a Θ-positive structure such that Θ-positive representations are precisely maximal representations. The only other classical family of Lie groups admitting a Θ-positive structure is given by $\mathrm{PO}(p, q)$.

The count of the connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{PO}(p, q))/\!\!/\mathrm{PO}(p, q)$ has been carried out, using techniques from the theory of Higgs bundles, by Aparicio-Arroyo, Bradlow, Collier, García-Prada, Gothen and Oliveira [ABC+18]. If the conjecture of Guichard–Labourie–Wienhard is true, their work also gives a parametrization of the space of Θ-positive representations in terms of holomorphic data. They show that exceptional components, only consisting of Zariski-dense representations, can only exist for the group $\mathrm{PO}(p, p+1)$. The $\mathrm{PO}(p, p+1)$-character variety admits $n(2g-2)-1$ connected components, previously parametrized by Collier, which conjecturally only consists of Zariski-dense representations [Col17]. These would be generalized Gothen components.

3. METRIC PROPERTIES OF THE ASSOCIATED LOCALLY SYMMETRIC SPACES: ANALOGIES AND DIFFERENCES WITH TEICHMÜLLER SPACE

In order to discuss some of the striking geometric properties of the locally symmetric spaces associated to Hitchin and maximal representations, we will need to discuss some more properties of the geometry of a symmetric space $\mathcal{X}$.

An important difference between higher rank symmetric spaces and their rank one relatives is that, in higher rank, the isometry group G does not act transitively on the unit tangent bundle, and a fundamental domain for the G-action on pairs of points in $\mathcal{X}$ is the Weyl chamber $\overline{\mathfrak{a}}^+$, which can be identified with a closed convex cone in a maximal flat subspace $\mathfrak{a}$ of $\mathcal{X}$. For example, in the case of the classical groups that will play a role in the sequel, we have

$$\begin{aligned} \overline{\mathfrak{a}}^+_{\mathrm{PSL}(n,\mathbf{R})} &= \{(\lambda_1, \ldots, \lambda_n) |\ \lambda_1 \geq \ldots \geq \lambda_n, \textstyle\sum \lambda_i = 0\} \\ \overline{\mathfrak{a}}^+_{\mathrm{PSp}(2n,\mathbf{R})} &= \{(\lambda_1, \ldots, \lambda_n) |\ \lambda_1 \geq \ldots \geq \lambda_n \geq 0\} \\ \overline{\mathfrak{a}}^+_{\mathrm{PO}(2,n)} &= \{(\lambda_1, \lambda_2) |\ \lambda_1 \geq \lambda_2 \geq 0\}. \end{aligned}$$

Parreau observed that it is possible to use the projection that associates to a pair of points in $\mathcal{X} \times \mathcal{X}$ the unique representative of their G-orbit in the Weyl chamber to define a vector valued distance $d^{\overline{\mathfrak{a}}^+} : \mathcal{X} \times \mathcal{X} \to \overline{\mathfrak{a}}^+$, which, she proves, satisfies a suitable triangular inequality. This is a *universal* distance in the sense that any G-invariant Finsler [6] distance d on $\mathcal{X}$ is induced by the composition of $d^{\overline{\mathfrak{a}}^+}$ with a suitable Weyl group invariant norm on $\mathfrak{a}$. As we will see, an important role in the study of higher rank Teichmüller theories will be played by the Finsler norms given by the symmetrized ℓ^∞ norm in the case of representations in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$ and by the ℓ^1 norm in the case of representations in $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(2n,\mathbf{R}))$; we will denote these two norms by ℓ^F. We then have

$$\text{(1)} \qquad \begin{array}{ll} \text{on } \overline{\mathfrak{a}}^+_{\mathrm{PSL}(n,\mathbf{R})} & \ell^F(\lambda_1,\ldots,\lambda_n) = \lambda_1 - \lambda_n \\ \text{on } \overline{\mathfrak{a}}^+_{\mathrm{PSp}(2n,\mathbf{R})} & \ell^F(\lambda_1,\ldots,\lambda_n) = \sum \lambda_i. \end{array}$$

Observe both these norms can be obtained as value of a linear functional which is always positive on the Weyl chamber. We will denote it by ϕ^F.

The Weyl chamber valued distance can be used to give a geometric interpretation of the Lyapunov and Cartan projections from the theory of Lie groups, which will be needed in the rest of the section. The Cartan projection, that we will denote by $\sigma : G \to \overline{\mathfrak{a}}^+$ depends on the choice of a maximal compact subgroup K of G (or equivalently of a point o in the symmetric space), and is induced by the Cartan decomposition: in the case of $G = \mathrm{PSL}_n(\mathbf{R})$, the vector $\sigma(g)$ is nothing but the ordered list of the logarithms of the singular values of the matrix g. We then have

$$\sigma(g) = d^{\overline{\mathfrak{a}}^+}(o, g \cdot o).$$

The Lyapunov projection $\lambda : G \to \overline{\mathfrak{a}}^+$ is induced by the Jordan decomposition of G: in the case of $G = \mathrm{PSL}_n(\mathbf{R})$, the vector $\lambda(g)$ is the ordered list of the logarithms of the absolute values of the eigenvalues of g. The Lyapunov projection can be geometrically reinterpreted as the translation length:

$$\lambda(g) = \inf_{x \in \mathcal{X}} d^{\overline{\mathfrak{a}}^+}(x, g \cdot x)$$

here the infimum can be understood as the vector of smaller Euclidean norm in the closure, and it is possible to prove that this is unique [Par12, Proposition 4.1].

3.1. Marked length spectra and compactifications

A lot of the geometry of a locally symmetric space associated to a representation ρ can be encoded in the *marked length spectrum* of ρ, the point in $(\overline{\mathfrak{a}}^+)^{\Gamma_g}$ that records, for every element γ in Γ_g, the Weyl chamber valued translation length $\lambda(\rho(\gamma))$. It is possible to verify that the map

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma_g, G)/\!\!/G & \to & (\overline{\mathfrak{a}}^+)^{\Gamma_g} \\ [\rho] & \mapsto & (\gamma \mapsto \lambda(\rho(\gamma))) \end{array}$$

is injective when restricted to a Hitchin or maximal component, and it is an interesting question to determine what is the minimal collection of lengths necessary to reconstruct

6. A *Finsler distance* on a smooth manifold $\mathcal{X}$ is the length function associated to a Finsler metric, a smooth choice of a, not necessarily Euclidean, norm on every tangent space.

the representation. Bridgeman, Canary and Labourie recently showed that, for Hitchin components, the knowledge of the absolute value of the top eigenvalue (the spectral radius) of all the elements in Γ_g corresponding to simple closed curves is enough [BCL17].

The projectivization of the marked length spectrum was used by Parreau to construct compactifications of the space of reductive representations of finitely generated groups in semisimple Lie groups [Par12]. In her work she interprets boundary points as projective classes of marked length spectra of actions on affine buildings: geometric objects that can be thought of as rescaled limits of the symmetric spaces. In affine buildings the curvature is replaced by branching and thus the geometry is encoded in a rich combinatorial structure.

We are just beginning to understand more detailed properties of such compactifications for higher rank Teichmüller theories. In [BP17] Burger and Pozzetti initiated the study of the actions on buildings arising in the compactifications of maximal representations, and Burger–Iozzi–Parreau–Pozzetti [BIPP19] described large domains of discontinuity for the mapping class group action on the boundary, a new phenomenon not present in Teichmüller theory. The combinatorial study of actions on buildings arising in the boundary of the Hitchin components was investigated by Martone [Mar18] using techniques inspired by the work of Fock and Goncharov, while the work of Kazarkov–Noll–Pandit–Simpson [KNPS17a, KNPS17b] aims at understanding the actions in Parreau's compactification from a more analytic point of view (cf. also the work of Collier–Li for a special class of degeneration [CL17]). Le [Le16], following ideas of Fock–Goncharov, used techniques of tropical geometry to interpret points in the boundary of the Hitchin component as higher laminations.

3.2. Parametrizations, Bers constants, Hamiltonian flows and entropy

Parametrizations provide a fundamental tool to construct examples of representations in the higher rank Teichmüller spaces, and explore finer geometric properties of the associated actions. These often generalise well known parametrizations of the classical Teichmüller space. Analogues of the shear coordinates on Teichmüller space have been developed in higher rank by Fock–Goncharov [FG06] in the case of Hitchin representations and by Alessandrini–Guichard–Rogozinnikov–Wienhard [AGRW] in the case of maximal representations. Analogues of Fenchel–Nielsen parametrization were developed by Bonahon–Dreyer [BD14] and Zhang [Zha15] for Hitchin representations and by Strubel [Str15] for maximal representations.

In all such parametrizations of Hitchin components new parameters associated to pairs of pants arise, the so-called *internal parameters*. These are not present in classical Teichmüller theory and account for many new higher rank phenomena. Most of these parametrizations are furthermore very concrete, and allow to compute examples of representations in higher rank Teichmüller thoeries with desired properties. For example, using Strubel's coordinates, Burger constructed examples of maximal representations with value in $\mathrm{PSp}(4,\mathbf{Z})$, in striking contrast with what happens in Teichmüller space (see also [LRT11] for some integral points in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$). In the rest of the subsection we will emphasize few other geometric features of Hitchn or maximal representations that have been also studied with the aid to suitable parametrizations.

The *Bers constant* is a universal constant C_g, depending on the genus g of S_g only, such that every hyperbolic structure on the surface S_g admits a pair of pants decomposition along curves of length bounded by C_g. This property of hyperbolizations has been of fundamental importance in the study of geometric properties of classical Teichmüller theory, for example in the construction of combinatorial models for the Teichmüller space [Bro02]. In his thesis [Zha15] Zhang used a parametrization of $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n, \mathbf{R}))$ inspired by Fenchel–Nielsen coordinates to show that this tool will not be available in higher rank Techmüller theory: Zhang constructs sequences of representations ρ_k in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n, \mathbf{R}))$ such that $\phi^F(\lambda(\rho_k(\gamma))) > k$ for every $\gamma \in \Gamma_g$, therefore no Bers constant can exist for Hitchin representations.

Suitable parametrizations also played an important role in the recent work of Sun–Wienhard–Zhang on the *symplectic geometry* of the Hitchin component. In his seminal paper [Gol84] Goldman constructed a symplectic form on character varieties of fundamental groups of surfaces, which, on the Teichmüller space, restricts to the Weil–Petersson symplectic form. In the case of the Hitchin component Sun–Wienhard–Zhang constructed a half dimensional family of commuting flows that are Hamiltonian for Goldman's symplectic form: these are associated to a pair of pants decomposition and are distinguished in two classes, the generalized twists along the curves in the decomposition, and the eruption flows, which only change the restriction of the representation to the fundamental group of the pair of pants, without changing the boundary holonomy [WZ18, SWZ17, SZ17]. The second kind of flows does not arise in classical Teichmüller theory, as there is a unique hyperbolic metric on a pair of pants with prescribed boundary holonomy.

Let $\phi : \overline{\mathfrak{a}}^+ \to \mathbf{R}^+$ be the restriction of a seminorm on $\mathfrak{a}$, or more generally a size function, as for example a positive linear functional. The *orbit growth rate*

$$h_\rho^{\sigma,\phi} := \lim_{T\to\infty} \frac{\log |\{\gamma \in \Gamma_g |\ \phi(\sigma(\rho(\gamma))) < T\}|}{T}$$

and the *entropy*

$$h_\rho^{\lambda,\phi} := \lim_{T\to\infty} \frac{\log |\{[\gamma] \in [\Gamma_g] |\ \phi(\lambda(\rho(\gamma))) < T\}|}{T}$$

are important invariants measuring the complexity of a locally symmetric space, and in particular of the locally symmetric spaces of the form $\rho(\Gamma_g)\backslash\mathcal{X}$ for a representation ρ in a higher rank Teichmüller theory. Here, as before, σ is the Cartan projection, and λ is the Lyapunov projection, while $[\Gamma_g]$ denotes the set of conjugacy classes in Γ_g. Both these quantities depend on the choice of a seminorm ϕ on $\overline{\mathfrak{a}}^+$; of particular interest are the seminorms induced by those linear functionals ϕ that are positive on $\lambda(\rho(\gamma))$ for every $\gamma \in \Gamma_g$. The orbit growth rate measures the exponential growth of the number of homotopy classes of loops based at a chosen base point and of ϕ-length bounded by T; on the other hand the entropy has a dynamical meaning as it can be reinterpreted, in many interesting cases, as the entropy of a suitable flow. Sambarino proved [Sam14] that orbit growth rate and entropy associated to a positive functional ϕ agree for representations in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$, or more generally for a suitable class of Anosov representations, but not much is known about the relations between these two invariants in full generality.

Using again a version of Fenchel–Nielsen parametrization of $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$ Zhang managed to construct sequences of representations ρ_k with $h^{\sigma,\phi^F}_{\rho_k} \to 0$ [Zha15] (cf. also [Nie15] where a similar result was proven for $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3,\mathbf{R}))$). This is again in strong contrast with the classical theory, in which the entropy is constant and equal to one, and along such degenerations most orbit points escape to infinity, a phenomenon that, in rank one, is prohibited by the compactness of the surface S_g. More recently Martone–Zhang [MZ16] proved, for both Hitchin and maximal representations, that the quantity $h^{\sigma,\phi^F}_{\rho}\mathrm{Syst}(\rho)$ is uniformly bounded away from zero and infinity with constants depending only on the genus of the surface; here $\mathrm{Syst}(\rho)$ is the panted systole, namely the length (with respect to the norm defined above) of the shortest curve not belonging to a pants decomposition of minimal length.

Using the thermodynamical formalism, and deep results of Sinai–Ruelle–Bowen, Potrie and Sambarino [PS17] proved that for every point $\rho \in \mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$ the entropy

$$h^{\lambda,\alpha_i}_{\rho} := \lim_{T\to\infty} \frac{\log |\{[\gamma] \in [\Gamma_g] |\ \alpha_i(\lambda(\rho(\gamma)) < T)\}|}{T}$$

is constant and equal to 1 for every simple root α_i, namely every linear functional $\alpha_i : \overline{\mathfrak{a}}^+ \to \mathbf{R}$ of the form $\alpha_i(\lambda(g)) = \lambda_i(g) - \lambda_{i+1}(g)$. Observe that this is not the restriction of a Weyl-invariant seminorm on $\mathfrak{a}$ and therefore it is not associated to a Finsler norm on $\mathcal{X}$. Nevertheless, as a consequence of this result, they deduce that the orbit growth rate with respect to the Riemannian metric is smaller or equal to 1 and equality characterises the Fuchsian locus, provided the metric is normalised so that the embedding of the hyperbolic plane equivariant with the principal $\mathrm{PSL}(2,\mathbf{R})$ has curvature -1.

3.3. Crossratios, identities, currents, and metrics

A fundamental tool in classical projective geometry is the projective crossratio on $\mathbf{FP}^1$ which extends the invariant of fourtuples in the affine chart $\mathbf{F} \subset \mathbf{FP}^1$ given by

$$b(x,y,z,t) = \frac{(z-x)(t-y)}{(y-x)(t-z)}.$$

This is the only invariant for the action of $\mathrm{PSL}(2,\mathbf{F})$ on distinct 4-tuples in $\mathbf{FP}^1$, and has been of fundamental importance both in projective geometry and hyperbolic geometry, through the identification $\mathbf{RP}^1 = \partial_\infty \mathbf{H}^2$.

Various other geometric structures on surfaces can also be understood using generalized crossratios on the boundary $\partial\Gamma_g$. Let $\partial\Gamma_g^{4*}$ denote the set of 4-tuples $(x,y,z,t) \in \partial\Gamma_g^4$

with $x \notin \{y,z\}$ and $t \notin \{y,z\}$. For our purposes [(7)], a crossratio will be a function

$$b : \partial\Gamma_g^{4*} \to \mathbf{R}$$

that is invariant under the diagonal Γ_g-action and satisfies the cocycle relations

$$\begin{aligned} b(x,y,z,w) &= b(x,y,t,w)b(x,t,z,w), \\ b(x,y,z,w) &= b(t,y,z,w)b(x,y,z,t). \end{aligned}$$

A crossratio is furthermore symmetric if $b(x,y,z,t) = b(z,t,x,y)$ and positive if $b(x,y,z,t) > 1$ for every positively oriented 4-tuple (x,y,z,t), all these properties are satisfied by the classical crossratio. Given an element $\gamma \in \Gamma_g$ its *period* for the crossratio b is

$$\mathrm{per}_b(\gamma) = \log|b(\gamma^-, x, \gamma \cdot x, \gamma^+)|$$

for any point x distinct from γ^+, γ^-. It is easy to check that the period doesn't depend on the choice of x. Observe that for the projective crossratio on $\mathbf{RP}^1 = \partial_\infty \mathbf{H}^2$, the period $\mathrm{per}_b(\gamma)$ equals the translation length of γ on $\mathbf{H}^2$.

Labourie [Lab07a] associated, to every representation in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n,\mathbf{R}))$ and $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(2n,\mathbf{R}))$, a crossratio whose periods are given by the translation lengths of the element $\rho(\gamma)$ with respect to the norm ℓ^F defined in (1). Hartnick–Strubel [HS12] extended Labourie's construction to maximal representations in any Hermitian Lie group; their normalisation is furthermore natural in the sense that whenever a Lie group homomorphism [(8)] $\eta : G \to H$ induces an inclusion $\eta_* : \mathrm{Max}(\Gamma_g, G) \subset \mathrm{Max}(\Gamma_g, H)$, the restriction of the crossratio of $\mathrm{Max}(\Gamma_g, H)$ is the crossratio of $\mathrm{Max}(\Gamma_g, G)$. More generally, for every $k = 1, \ldots, \lfloor n/2 \rfloor$ Martone and Zhang [MZ16] associated, to any Hitchin representation, a symmetric crossratio whose periods are given by

$$\sum_{i=1}^{k} \log(\lambda_i(\rho(\gamma))) - \log(\lambda_{n-i}(\rho(\gamma))).$$

In the case of maximal representations Burger–Pozzetti [BP17] also studied a vector valued generalization of such crossratios, which allows to study finer geometric properties of the symmetric spaces associated to such representations.

These crossratios are a starting point to obtain generalisations, to higher rank Teichmüller spaces, of various beautiful *identities* between lengths of curves, which were previously known for hyperbolic surfaces. Labourie–McShane [LM09] proved McShane-type identities for Hitchin and maximal representations (see also [HS19] for a more recent approach), the Basmajian identity was generalized by Vlamis–Yarmola [VY17] to Hitchin representations and by Fanoni–Pozzetti for maximal representations [FP16]. Strictly speaking all the aforementioned identities hold for surfaces with boundary, which therefore are not encompassed in Definition 2.1. However, on the one hand it is possible

7. The reader should be warned that there are various different pairwise not equivalent conventions in the definition of a crossratio [Led95, Ham97, Lab07a, MZ16, Bey17], we adopted here the order convention of [BP17].

8. Such Lie group homomorphisms are precisely the *tight embeddings* defined and studied in [BIW09].

to define higher rank Teichmüller theories for surfaces with boundary [FG06, BIW10], on the other hand the identities we just discussed give interesting corollaries also for compact connected surfaces, when one considers the restriction of the representation to subsurfaces with boundary.

Another important application of the theory of crossratios for higher rank Teichmüller spaces is the construction of *geodesic currents* associated to representations in these components. Geodesic currents $\mathcal{C}(S_g)$ are Γ_g-invariant Radon measures on the space of unoriented, unparametrized geodesics in $\widetilde{S}_g$; these objects were introduced by Bonahon [Bon88] in the study of the geometry of hyperbolic surfaces, as this theory includes both hyperbolic structures and closed geodesics on surfaces: to a closed geodesic $\gamma \subset S_g$ corresponds the geodesic current δ_γ which is the sum of a Dirac mass on each lift of γ to $\widetilde{S}_g$. A key feature of the theory of geodesic currents is that the geometric intersection of closed geodesics extends to a bilinear form $i : \mathcal{C}(S_g) \times \mathcal{C}(S_g) \to \mathbf{R}$ which encodes a lot of the geometry of the surface.

To every continuous symmetric positive crossratio b one can naturally associate a Liouville geodesic current μ_b, whose intersection with simple closed curves gives the period of the crossratio: $i(\mu_b, \delta_\gamma) = \mathrm{per}_b(\gamma)$. Note that this last property completely characterize the current μ_b. As a result higher rank Teichmüller theories are part of the theory of geodesic currents [MZ16]. An important consequence of this fact is that the length functions associated to maximal representations satisfy a length shortening under surgery property: if a closed geodesic representing an element γ has a self intersection point $x \in S_g$ and $\gamma = \gamma_1\gamma_2 \in \pi_1(S_g, x)$ where γ_i are the two sub-paths of γ beginning and ending at x, then $\ell^F(\lambda(\rho(\gamma_1))) + \ell^F(\lambda(\rho(\gamma_2))) \leq \ell^F(\lambda(\rho(\gamma)))$ and $\ell^F(\lambda(\rho(\gamma_1\gamma_2^{-1}))) \leq \ell^F(\lambda(\rho(\gamma)))$ [MZ16, Proposition 4.5]. This is one of the many features showing that representation in higher rank Teichmüller theories remember a lot of the topology of the underlying surface.

Using the thermodynamical formalism, Bridgeman–Canary–Labourie–Sambarino defined Riemannian metrics on Hitchin and maximal components (and more generally on spaces of Anosov representations), the so-called *pressure metric* [BCLS15]. It is obtained as the Hessian of the renormalized intersection

$$J^\phi(\rho, \eta) = \frac{h_\rho^{\lambda,\phi}}{h_\eta^{\lambda,\phi}} \lim_{T\to\infty} \frac{1}{|L_\rho^\phi(T)|} \sum_{\gamma \in L_\rho^\phi(T)} \frac{\phi(\lambda(\eta(\gamma)))}{\phi(\lambda(\rho(\gamma)))}$$

where ϕ is the maximum eigenvalue in the case of $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(n, \mathbf{R}))$ while it is the sum of the eigenvalues for $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(2n, \mathbf{R}))$, and $L_\rho^\phi(T)$ denotes the set of conjugacy classes in Γ_g for which $\phi(\lambda(\rho(\gamma))) < T$. They can show that the restriction of the pressure metric to the Fuchsian locus agrees with the Weil–Petersson metric. Labourie–Wentworth [LW18] combined holomorphic and dynamical techniques to compute an expression for the pressure metric on the Fuchsian locus in the Hitchin component; in the same article they generalise Gardiner's formula for the translation length of closed geodesics to a variational formula of the p-th largest eigenvalue of the holonomy of a Hitchin representation along a simple closed geodesic.

More recently, Brigeman–Canary–Labourie–Sambarino defined a new metric on the Hitchin component, the so-called Liouville pressure metric [BCLS18], which is obtained as a renormalized intersection with respect to the first root $\alpha_1(\lambda) = \lambda_1 - \lambda_2$. Since $h_\rho^{\lambda,\alpha_1} = 1$ [PS17], no normalization is required for this new intersection function; furthermore they reinterpret the intersection $J^\phi(\rho, \eta)$ as a ratio of pairings of the non-symmetrized Liouville currents with the simple root flow associated to the two Hitchin representations. Bridgeman–Pozzetti–Sambarino–Wienhard pointed out that the Liouville pressure metric has the additional property that, as the Weil–Petersson metric on Teichmüller space, it can be reinterpreted as the Hessian of the Hausdorff dimension in purely imaginary directions, when $\mathrm{Hit}(\Gamma_g, G)$ is considered as a subspace of the character variety in the complexified group $G^{\mathbf{C}}$ [PSW19].

3.4. Collar lemma

A fundamental geometric property of hyperbolic surfaces is given by the collar lemma, first established by Keen [Kee74]. It states that any simple closed geodesic in a hyperbolic surface admits an embedded collar whose width can be explicitly determined as a function of the length of the geodesic, and diverges logarithmically as the length shrinks to zero. In particular this can be used to obtain a lower bound on the length of any geodesic intersecting a simple geodesic β in terms of the length of β. Surprisingly, the same holds for Hitchin and maximal representations, although the sets of minimal displacement of two elements $\rho(\alpha), \rho(\beta)$ corresponding to intersecting simple closed curves need not intersect in the symmetric space:

Theorem 3.1 (Lee–Zhang [LZ17]). — *Let $\rho : \Gamma_g \to \mathrm{PSL}(n, \mathbf{R})$ be a Hitchin representation, and let $\alpha, \beta \in \Gamma_g$ be such that the axis of the corresponding isometries intersect. Then for every $k = 0, \ldots, n-2$ it holds*

$$\frac{\lambda_1(\rho(\alpha))}{\lambda_n(\rho(\alpha))} \geq \frac{\lambda_k(\rho(\beta))}{\lambda_k(\rho(\beta)) - \lambda_{k+1}(\rho(\beta))}.$$

A similar result for maximal representations was proven, with different techniques, by Burger–Pozzetti [BP17], and shows that a lot of the topology of the surface S_g is reflected in the geometry of locally symmetric spaces associated to representations in higher rank Teichmüller theories. It is worth remarking that both the collar lemma for Hitchin representations and the one for maximal representations relate the Finsler length of one element with the logarithm of the eigenvalue gap of the other, and thus measure a finer geometric property than what can be seen by the crossratios discussed above.

4. MINIMAL SURFACES

An important analytic tool to study higher rank Teichmüller theories are harmonic maps and minimal surfaces, a standard reference for this is [SU82]. As the Zariski closure of any Hitchin or maximal representation is a reductive group, for every choice of a conformal structure Σ on the surface S_g there exists a unique equivariant harmonic map $f : \widetilde{S}_g \to \mathcal{X}$: this is a minimizer of the Dirichlet energy

$$\int_\Sigma \|df\|^2 dV.$$

The harmonic map f crucially depends on extrinsic choice of the conformal structure Σ, and therefore it is natural to look for a preferred choice of a complex structure, which should reflect better the properties of the locally symmetric space associated to the representation ρ.

A *branched minimal immersion* is a harmonic map f that is furthermore weakly conformal; in this case it is also an area minimizer. It follows from the properness of the mapping class group action on the Hitchin and maximal representations [Lab08] that minimal harmonic maps always exist. Conversely Labourie conjectured uniqueness for such a map:

Conjecture 4.1 (Labourie). — *For every $\rho \in \mathrm{Hit}(\Gamma_g, G)$ there exists a unique ρ-equivariant minimal immersion $f : \widetilde{S}_g \to \mathcal{X}$.*

A positive answer to Labourie's conjecture would be of fundamental interest for a number of reasons. In particular, using the theory of Higgs bundles, it would allow to obtain a mapping class group equivariant parametrization of the associated higher rank Teichmüller theories in terms of holomorphic data. A second important reason emerged from Collier–Alessandrini's work [AC17]: they showed that, on every component on which Labourie's conjecture holds, there is a natural complex structure which is, again, equivariant for the mapping class group action.

Significant progress on Labourie's conjecture has only been achieved in rank 2: for $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$ the uniqueness of minimal harmonic maps was obtained independently by Labourie [Lab97, Lab07b] and Loftin [Lof01]. More recently, also building on ideas of Baraglia about cyclic Higgs bundles [Bar15], Labourie developed [Lab17] the theory of cyclic surfaces that he used to give a unified affirmative solution to the uniqueness of minimal harmonic maps for $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$, $\mathrm{Hit}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$ and $\mathrm{Hit}(\Gamma_g, \mathrm{G}_2)$. The same techniques were then generalized by Collier [Col16] to the Gothen components in $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$, and by Alessandrini–Collier for the other components in $\mathrm{Max}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$ [AC17]. Using completely different techniques, which we will partially discuss in the last section of the survey, the validity of the conjecture has been established for $\mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n))$ by Collier–Tholozan–Toulisse [CTT17].

5. RELATIONS WITH GEOMETRIC STRUCTURES

Given a Lie group G and a model manifold X on which G acts transitively and effectively, a (G, X)-geometric structure on a manifold M of dimension $\dim(M) = \dim(X)$ is the datum of an atlas of M with image in X so that the transition functions are elements of G. Every (G, X) structure is determined by its associated holonomy $\rho : \pi_1(M) \to G$ and developing map dev : $\widetilde{M} \to X$. As semisimple Lie groups G act on various homogeneous manifolds X, we will see that we can often reinterpret representations in higher rank Teichmüller theories, and more generally Anosov representations, as holonomies of geometric structures on non-necessarily compact manifolds.

Given a representation $\rho : \Gamma_g \to G$, Guichard–Wienhard [GW12] constructed the first examples of domains of proper discontinuity Ω_ρ for the action of $\rho(\Gamma_g)$ on G/P where P is a suitable parabolic subgroup. They observed that it is in most cases possible to choose a parabolic subgroup P, so that, after removing a closed subset K_ξ determined by the image of the ρ-equivariant boundary map ξ, the action on the complement $\Omega_\rho = (G/P) \setminus K_\xi$ is properly discontinuous; for many choices of P, even if not all, such action is also cocompact. The representation ρ can then be re-interpreted as the holonomy of a $(G, G/P)$-structure on the manifold $\Omega_\rho/\rho(\Gamma_g)$. In more recent work Kapovich–Leeb–Porti gave a general criterion to construct a much more general class of domains of discontinuity, where the set K_ξ depends on the choice of a balanced ideal in the Weyl group [KLP18]; Stecker proved [Ste18] that for Hitchin representations every cocompact domain of discontinuity in the full flag manifold $\mathcal{F}$ is one of the domains constructed by Kapovich–Leeb–Porti. The even richer theory of domains of discontinuity in oriented flag manifolds has been developed by Stecker–Treib [ST18]. Dumas–Sanders conjecture that the set Ω_ρ is, in these cases, a fiber bundle over the surface S_g with compact fiber [DS17]. This conjecture has been settled by Alessandrini–Li [AL] for representations in the Hitchin component provided the rank is smaller than 63.

In general a problem in this interpretation of higher rank Teichmüller theories as holonomies of (G, X)-structures is that it is not always easy to characterise what geometric structures arise with this construction, nor the topology of the quotient manifold M [9]. In some specific cases, and in particular in low ranks, it is however possible to use harmonic maps, and more generally the theory of Higgs bundles to give a more precise answer to some of these questions. We will discuss, in the next sections, some geometric implications of this approach, but we refer the reader to the survey [Ale18] for details about the Higgs bundles perspective on these same themes.

5.1. Convex projective structures

Convex projective structures are an important example of geometric structures: these are $(\mathrm{PSL}(d+1, \mathbf{R}), \mathbf{RP}^d)$-structures on a manifold M with the additional property that there exists a compatible identification of the universal covering $\widetilde{M}$ with an open subset Ω_ρ contained in an affine chart of $\mathbf{RP}^d$. In other terms the developing map dev : $\widetilde{M} \to \mathbf{RP}^d$ is injective and its image is contained in an affine chart and is convex therein. If we furthermore require M to be compact, then M can be homeomorphic to S_g only if $d = 2$. In this case Choi–Goldman [CG93] proved that the Hitchin

9. Dumas–Sanders compute the homology of the quotient of the associated domain of discontinuity in $G^{\mathbf{C}}/P^{\mathbf{C}}$.

component $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$ bijectively corresponds to the parameter space of convex projective structures on S_g. Baraglia, in his thesis [Bar10], used Higgs bundles, together with the solution, due to Labourie and Loftin, of Labourie's conjecture to give an analytic construction of the convex projective structure associated to a representation $\rho \in \mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$. The interpretation of the Hitchin component as parameter space for convex projective structures on S_g gives further geometric significance to the norm ℓ^F introduced at the beginning of Section 3: every properly convex projective domain Ω_ρ is endowed with a Finsler distance d^H, the *Hilbert metric*, which is invariant under the group of projective automorphisms of Ω_ρ. The value $\ell^F(\lambda(\rho(\gamma)))$ is nothing but the translation length of $\rho(\gamma)$ on (Ω_ρ, d^H).

Improving on work of Benoist–Hulin [BH14], Tholozan [Tho17] gave a precise relation of the Hilbert metric with the Blaschke metric, another invariant metric on a properly convex projective domain, whose analytic definition arises from the theory of affine spheres. Such comparison allowed him to deduce that for every representation $\rho \in \mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$ there exists a representation $\eta \in \mathrm{Hit}(\Gamma_g, \mathrm{PSL}(2, \mathbf{R})) = \mathcal{T}(S_g)$ which is strictly dominated by ρ in the sense that there exists a constant $K > 1$ such that for every $\gamma \in \Gamma_g$, $\phi^H(\lambda(\rho(\gamma))) > K\phi^H(\lambda(\eta(\gamma)))$. Here the right hand side is nothing but the translation length in $\mathbf{H}^2$.

More generally, Danciger–Guéritaud–Kassel [DGK17] and Zimmer [Zim17] independently proved that Hitchin representations in odd dimensions are holonomies of convex projective structures on non-compact manifolds, which are however convex cocompact. Again $\phi^F(\lambda(\rho(\gamma)))$ can be reinterpreted as the translation length for the Hilbert metric in these manifolds. Similarly Danciger–Guéritaud–Kassel show that every $\rho \in \mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n))$ is the holonomy of a convex projective structure on a non-compact manifold which is convex cocompact [DGK18].

5.2. Properly convex foliated projective contact structures

In the case of $\mathrm{Hit}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$ Guichard and Wienhard [GW08] interpreted the quotient of the domain of discontinuity $\Omega_\rho \subset \mathbf{RP}^3$ as a (marked) properly convex foliated projective contact structure on the unit tangent bundle T^1S_g and proved that the Hitchin component can be reinterpreted as the moduli space of such structures. Here a *projective contact structure* is a $(\mathrm{PSp}(4, \mathbf{R}), \mathbf{RP}^3)$-geometric structure and this refers to the fact that $\mathbf{RP}^3$ admits a contact structure invariant for the $\mathrm{PSp}(4, \mathbf{R})$-action which is thus inherited by the geometric structure on M; properly convex refers to the fact that, while the full image of $\mathrm{dev} : T^1\widetilde{S}_g \to \mathbf{RP}^3$ will not be convex, when $T^1\widetilde{S}_g$ is identified with $\mathrm{PSL}(2, \mathbf{R})$, the image, under the developing map, of every parabolic subgroup is required to be a properly convex subspace of $\mathbf{RP}^3$.

In his thesis Baraglia [Bar10] gave a different construction of these projective structures developing a good understanding of the geometric properties of the Higgs bundle associated to a representation in $\mathrm{Hit}(\Gamma_g, \mathrm{PSp}(4, \mathbf{R}))$; using such analytic input he found a surprising bridge with different geometric structures. To be more precise, he used the

Higgs bundles theory to associate to any ρ-equivariant harmonic map $f^{\mathcal{X}} : \widetilde{S}_g \to \mathcal{X}$ with values in the symmetric space $\mathcal{X}$, a map $f^Q : \widetilde{S}_g \to Q$ with values in the Klein quartic Q, which is nothing but the Plücker embedding of the Grassmannian $\mathrm{Gr}_2(\mathbf{R}^4)$ in $\mathbf{RP}^5$. Thus every point $q \in Q$ corresponds on the one hand to a 2-dimensional subspace of $\mathbf{R}^4$, or equivalently to a projective line $\mathbf{RP}^1 \subset \mathbf{RP}^3$, and on the other hand to the projective class of an elementary vector $v \wedge w$.

When regarded as a subset of $\mathbf{RP}^5$, the Klein quartic Q can be identified with the set of isotropic vectors for the canonical bilinear form b on $\mathbf{R}^6 = \wedge^2\mathbf{R}^4$: this is defined by requiring that $v \wedge w \wedge z \wedge t = b(v \wedge w, z \wedge t) e_1 \wedge e_2 \wedge e_3 \wedge e_4$, and is clearly preserved by the induced action of $\mathrm{PSp}(4, \mathbf{R}) \subset \mathrm{PSL}(4, \mathbf{R})$. Such metric has signature $(3, 3)$. We thus get that $Q = \mathrm{Is}(\mathbf{R}^{3,3})$ inherits a conformal class of pseudo-Riemannian metrics of signature $(2, 2)$. Baraglia shows that the map $f^Q : \widetilde{S}_g \to Q$ is space-like, namely the restriction of the conformal class of metrics to any tangent plane is positive definite. He deduces from this that the projective lines associated to any pair of distinct points in the image of f^Q are disjoint and contained in the domain of discontinuity Ω_ρ, which therefore can be identified with the unit tangent bundle $T^1\widetilde{S}_g$. Furthermore Baraglia shows that the fibers of the induced projection $\Omega_\rho \to \widetilde{S}_g$, are transverse to the contact distribution on $\mathbf{RP}^3$ induced by the symplectic structure on $\mathbf{R}^4$; along the way he also describes how to use the map f^Q to construct a ρ-equivariant minimal immersion in the pseudo-Riemannian symmetric space $\mathbf{H}^{2,3}$.

More generally, both in Guichard–Wienhard [GW08] and Baraglia's work [Bar10] it is possible to find a geometric interpretation of the Hitchin component $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(4, \mathbf{R}))$ as moduli space of properly convex foliated projective structures. Ideas similar to the ones developed by Baraglia were used more recently in the work of Alessandrini–Li [AL18] to understand a class of representations in $\mathrm{PO}_0(2, 2)$ giving rise to compact Anti-de Sitter 3-manifolds. These representations, however, do not belong to a higher rank Teichmüller theory.

5.3. The geometry of maximal representations in $\mathrm{PO}_0(2, n)$

Collier–Tholozan–Toulisse [CTT17] recently obtained a major generalization Baraglia's work outlined in the previous section for $\mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n+1))$. Again, combining Higgs bundles techniques with the study of different geometric structures, they gave a precise answer to most of the questions outlined in sections 2 and 3.

Let $\mathbf{R}^{2,n+1}$ denote a real vector space endowed with a bilinear form h of signature $(2, n+1)$ preserved by the group $\mathrm{PO}_0(2, n+1)$. In Collier–Tholozan–Toulisse's work four different homogeneous $\mathrm{PO}_0(2, n+1)$-spaces play an important role: the symmetric space $\mathcal{X} = \mathrm{PO}_0(2, n+1)/\mathrm{P}(\mathrm{O}(2) \times \mathrm{O}(n+1))$ which can be identified with the set of two dimensional subspaces of $\mathbf{R}^{2,n+1}$ on which h is positive definite, the pseudo-Riemannian symmetric space $\mathbf{H}^{2,n} = \mathrm{PO}_0(2, n+1)/\mathrm{PO}_0(2, n)$ which corresponds to the set of negative definite lines in $\mathbf{P}(\mathbf{R}^{2,n+1})$, as well as the two maximal parabolic quotients of $\mathrm{PO}_0(2, n+1)$: the photon space $\mathrm{Pho}(\mathbf{R}^{2,n+1})$, namely the set of isotropic planes in $\mathrm{Gr}_2(\mathbf{R}^{2,n+1})$, and the Einstein universe $\mathrm{Ein}(1,n)$ which coincides with the set of isotropic lines in $\mathbf{P}(\mathbf{R}^{2,n+1})$; this is the Shilov boundary of the Hermitian symmetric space $\mathcal{X}$.

Let now $\rho \in \mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n+1))$. A ρ-equivariant map $f^{2,n} : \widetilde{S}_g \to \mathbf{H}^{2,n}$ is a *space-like embedding* if the restriction of h to the tangent plane to $f(x)$ is positive definite for every $x \in \widetilde{S}_g$. In this case f is canonically associated, via the Gauss map, to a ρ-equivariant map $f^{\mathcal{X}} : \widetilde{S}_g \to \mathcal{X}$, simply defined as $f^{\mathcal{X}}(x) := [df^{2,n}(T_x\widetilde{S}_g)] \in \mathcal{X}$. In order to prove Labourie's conjecture for ρ (cf. Section 4), Collier–Tholozan–Toulisse observe that the Dirichlet energy can also be defined for space-like embeddings $f^{2,n}$ and, using Higgs bundles, prove that the critical points for the functional, namely the maximal space-like embeddings $f^{2,n}$, correspond bijectively, via the Gauss map, to minimal harmonic maps $f^{\mathcal{X}}$. They then prove the uniqueness of maximal space-like embeddings in $\mathbf{H}^{2,n}$ using techniques from pseudo-Riemannian geometry.

Interestingly, for most representations $\rho \in \mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n+1))$, the group Γ_g cannot act properly discontinuously on the whole $\mathbf{H}^{2,n}$ but it will admit a domain of discontinuity $\Omega_\rho^{\mathbf{H}^{2,n}}$. As was the case in Section 5.2 for representations in $\mathrm{Hit}(\Gamma_g, \mathrm{PSL}(3, \mathbf{R}))$, the length function ℓ^F that we associated, in Section 3, to the maximal representation ρ, can be reinterpreted, in this context, as a generalized translation distance in the pseudo-Riemannian setting. Maximal space-like embeddings always exist, and Collier–Tholozan–Toulisse observe that, by their very definition, the curvature of these embeddings is always bounded above by -1; as a consequence they obtain that for every representation $\rho \in \mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n+1))$ that doesn't belong to the Fuchsian locus, there exists a representation $\eta \in \mathcal{T}(S_g)$ which is strictly dominated by ρ in the sense that there exists a constant $K > 1$ such that for every $\gamma \in \Gamma_g$, $\phi^F(\lambda(\rho(\gamma))) > K\phi^F(\lambda(\eta(\gamma)))$. Furthermore η can be chosen to be the hyperbolic structure whose conformal structure provides the solution to Labourie's conjecture. This gives a very strong comparison between the length function associated to a maximal representations and the ones associated to hyperbolic structures, and allows, for example, in this cases, to give a different proof of the result of Potrie–Sambarino recalled at the end of Section 3.2, as well as of different versions of the identities from Section 3.3 and of the collar lemma from Section 3.4.

Another important application of the theory of maximal, ρ-equivariant, space-like surfaces in $\mathbf{H}^{2,n}$ is a precise description of the topology of the quotient of the domain of discontinuity for the ρ-action on $\mathrm{Pho}(\mathbf{R}^{2,n+1})$, as well as a characterization of which geometric structures arise this way: in agreement with the conjecture of Dumas–Sander, the quotient of the domain is a fiber bundle over the surface S_g with fiber $\mathrm{Pho}(\mathbf{R}^{2,n})$. In order to prove this last result, Collier–Tholozan–Toulisse use again the unique maximal space-like embedding $f^{2,n}$, and observe that to every point x in the image of $f^{2,n}$, the restriction of h to the orthogonal $x^\perp$ has signature $(2, n)$ and thus there is a naturally associated subspace $\mathrm{Pho}(x^\perp) \subset \mathrm{Pho}(\mathbf{R}^{2,n+1})$. In order to conclude, they observe that, as the surface is space-like, for every pair of distinct points x, y the subspaces $\mathrm{Pho}(x^\perp)$,

$\mathrm{Pho}(y^\perp)$ are disjoint and contained in the domain of discontinuity $\mathrm{Pho}(\mathbf{R}^{2,n+1}) \setminus K_\xi$. Their result is much more precise than what we discussed here: not only they manage to explicitly describe the topology of the bundle in terms of holomorphic data associated to the representation, but they also show that any fibered photon structure arises with this construction, thus giving an interpretation of the higher rank Teichmüller spaces $\mathrm{Max}(\Gamma_g, \mathrm{PO}_0(2, n+1))$ as parameter spaces of concrete geometric structures on explicit compact manifolds.

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